Daubechies wavelets

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The purpose of the Julia package IntervalWavelets.jl is to compute ordinary Daubechies scaling functions (see e.g. [2]) and the moment-preserving boundary scaling functions from [1]. We rely on the recursive approach of [3]. In this note I summarize these methods with all the details used in the implementation.

1 Interior scaling functions

A Daubechies scaling function ϕ and associated wavelet ψ with p vanishing moments are defined by a filter $\{h_k\}_{k\in\mathbb{Z}}$. The filter, scaling function and wavelet have supports of the same lengths and we know from [2, Theorem 7.5] that if $\operatorname{supp}\{h_k\}_k = [N_1, N_2]$, then

$$\operatorname{supp}\phi = [N_1, N_2], \quad \operatorname{supp}\psi = \Big[\frac{N_1 - N_2 + 1}{2}, \frac{N_2 - N_1 + 1}{2}\Big].$$

It is customary to let $N_1 = 0$ and $N_2 = 2p-1$. However, when constructing the boundary scaling functions we have $N_1 = -p+1$ and $N_2 = p$.

The scaling function satisfies the dilation equation

$$\phi(x) = \sqrt{2}\sum_{k=0}^{2p-1} h_k\phi_k(2x) = \sqrt{2}\sum_{k=0}^{2p-1} h_k\phi(2x-k). \tag{1}$$

For $p \geq 2$, ϕ is continuous and hence zero at the endpoints of the support. These properties allow us to compute ϕ at the integer values (in the support). As an example, for $p = 3$:

$$\begin{aligned}
\phi(1) &= \sqrt{2}\big(h_1\phi(1) + h_0\phi(2)\big),\\
\phi(2) &= \sqrt{2}\big(h_4\phi(1) + h_3\phi(2) + h_2\phi(3) + h_1\phi(4)\big),\\
\phi(3) &= \sqrt{2}\big(h_5\phi(1) + h_4\phi(2) + h_3\phi(3) + h_2\phi(4)\big),\\
\phi(4) &= \sqrt{2}\big(h_5\phi(3) + h_4\phi(4)\big).
\end{aligned}$$

In matrix form, we have an eigenvalue problem:

$$\begin{bmatrix}\phi(1)\\ \phi(2)\\ \phi(3)\\ \phi(4)\end{bmatrix} = \sqrt{2}\begin{bmatrix}h_1 & h_0 & 0 & 0\\ h_3 & h_2 & h_1 & h_0\\ h_5 & h_4 & h_3 & h_2\\ 0 & 0 & h_5 & h_4\end{bmatrix}\begin{bmatrix}\phi(1)\\ \phi(2)\\ \phi(3)\\ \phi(4)\end{bmatrix}.$$

For a general support $[N_1, N_2]$ the (i,j)'th entry is $\sqrt{2}h_{2i-j+N_1}$ and the vector $\boldsymbol{\phi} = [\phi(n)]_{n=1}^4$ is an eigenvector of the eigenvalue 1. This eigenspace is one-dimensional, so the only question is how to scale $\boldsymbol{\phi}$. From e.g. [1, page 69] we know that

$$\sum_{k\in\mathbb{Z}} \phi(k) = \sum_{k=0}^{2p-1} \phi(k) = 1.$$

From the function values at the integers we can compute the function values at the half-integers using (1). As an example,

$$\phi\Big(\frac{3}{2}\Big) = \sqrt{2}\big(h_0\phi(3) + h_1\phi(2) + h_2\phi(1)\big).$$

This process can be repeated to recursively yield $\phi(k/2^R)$, for all integers k and positive integers R.

Note that the filter $\{h_k\}_k$ defining the scaling function is not unique. In fig. 1 is shown the usual, minimum-phase Daubechies 4 scaling function along with Daubechies 'symmlet'/linear phase scaling function used in section 2 and [1] – see e.g. [2, Section 7.2.3].

2 Boundary scaling functions

The moment preserving Daubechies boundary scaling functions were introduced in [1] and are also described in [2] (albeit with some indexing errors).

An important difference between the internal and boundary scaling functions is that the left (right) boundary scaling functions are *not* continuous at the left (right) endpoint of their support.

As in section 1, the dilation equations defining the boundary scaling functions can yield function values at all dyadic rationals once we have the function values at the integers (in the support). In the subsequent sections the focus is therefore on how to compute these functions at the integers.

The filters used for the boundary scaling functions are available at `https://services.math.duke.edu/~ingrid/publications/54.txt` and `http://numerical.recipes/contrib`.

2.1 Left boundary scaling functions

Let p denote the number of vanishing moments and ϕ be the interior symmlet Daubechies scaling function associated with the wavelet with p vanishing moments translated such that $\operatorname{supp}\phi = [-p+1, p]$.

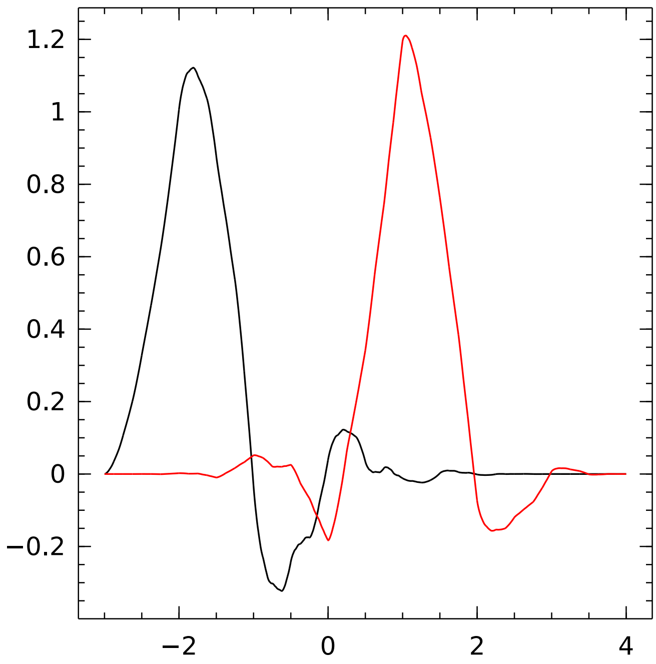

Figure 1: The usual minimum-phase Daubechies 4 scaling function (black) and the symmlet version (red).

We want a family of functions satisfying a multiresolution analysis on $L^2([0,\infty))$ or, equivalently, a dilation equation like eq. (1). The starting point is $\{\phi_k\}_{k\geq 0}$. The functions ϕ_k with $\operatorname{supp}\phi_k \subseteq [0,\infty)$ do not need any alteration. But the ϕ_k with $\operatorname{supp}\phi_k \cap (-\infty, 0) \neq \emptyset$ (i.e., with $0 \leq k < p-1$) must be replaced with a corresponding ϕ_k^{left}. It turns out that we should also replace ϕ_{p-1} with ϕ_{p-1}^{left} in order to keep the number of vanishing moments even though $\operatorname{supp}\phi_{p-1} = [0, 2p-1]$. The boundary scaling functions are constructed such that $\operatorname{supp}\big(\phi_k^{\text{left}}\big) = [0, p+k]$.

The relevant counterpart to the dilation equation eq. (1) for interior scaling functions is

$$\begin{aligned}\phi_k^{\text{left}}(x) &= \sqrt{2}\sum_{l=0}^{p-1} H_{k,l}^{\text{left}}\phi_l^{\text{left}}(2x) + \sqrt{2}\sum_{m=p}^{p+2k} h_{k,m}^{\text{left}}\phi_m(2x) \\ &= \sqrt{2}\sum_{l=0}^{p-1} H_{k,l}^{\text{left}}\phi_l^{\text{left}}(2x) + \sqrt{2}\sum_{m=p}^{p+2k} h_{k,m}^{\text{left}}\phi(2x-m), \end{aligned} \tag{2}$$

for $0 \leq k \leq p-1$, where $(H_{k,l}^{\text{left}})$ and $(h_{k,m}^{\text{left}})$ are filter coefficients.

For $x \neq 0$ we make use of the compact support. Consider e.g. the case $p = 2$ (where is ϕ is supported on $[-1,2]$, ϕ_0^{left} is supported on $[0,2]$ and ϕ_1^{left} is supported on $[0,3]$):

$$\begin{aligned}\phi_0^{\text{left}}(1) &= \sqrt{2}\big(H_{0,1}^{\text{left}}\phi_1^{\text{left}}(2) + h_{0,2}^{\text{left}}\phi(0)\big), \\ \phi_1^{\text{left}}(1) &= \sqrt{2}\big(H_{1,1}^{\text{left}}\phi_1^{\text{left}}(2) + h_{1,2}^{\text{left}}\phi(0)\big), \\ \phi_1^{\text{left}}(2) &= \sqrt{2}\big(h_{1,3}^{\text{left}}\phi(1) + h_{1,4}^{\text{left}}\phi(0)\big).\end{aligned}$$

From Section 1 we know how to calculate the internal scaling function and by starting with the largest values we can also compute the boundary scaling function.

The function value $\phi_k^{\text{left}}(0)$ require special treatment. For $x = 0$ (2) becomes

$$\phi_k^{\text{left}}(0) = \sqrt{2}\sum_{l=0}^{p-1} H_{k,l}^{\text{left}}\phi_l^{\text{left}}(0).$$

These function values are therefore an eigenvector of the matrix with (i,j)'th entry $[\sqrt{2}H_{i,j}^{\text{left}}]$. To find the proper normalization of this eigenvector we can do as follows: Let $\phi_\ell^{\text{left}}(0) = y_\ell$ for $0 \leq \ell < p$. Computing an eigenvector as described we have $\phi_\ell^{\text{left}}(0) = z_\ell = c y_\ell$ for some $c \in \mathbb{R}$. We know that the left boundary scaling functions are a basis capable of reconstructing polynomials. In particular, there exists $a_0, \dots, a_{p-1}$ such that for all $x \in [0,1]$ we have

$$\sum_{\ell=0}^{p-1} a_\ell \phi_\ell^{\text{left}}(x) = 1.$$

On this interval all the interior scaling functions are zero and we only need the left boundary scaling functions. The coefficients can be determined by choosing p dyadic rationals $x_0, \dots, x_{p-1}$ with $0 < x_k \leq 1$ and solving the linear equations

$$\sum_{\ell=0}^{p-1} a_\ell \phi_\ell^{\text{left}}(x_k) = 1, \quad 0 \leq k < p.$$

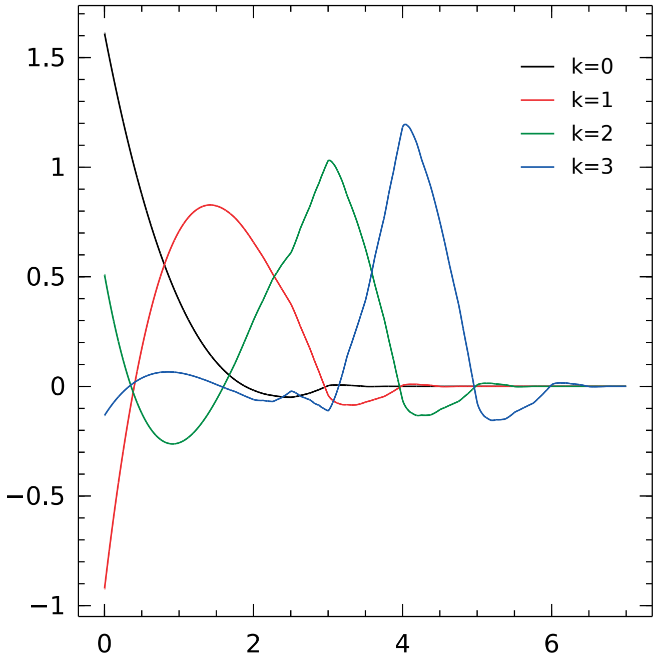

Figure 2: The left boundary scaling function with 4 vanishing moments.

In order to have have the p function values $\phi_\ell^{\text{left}}(x_k)$ the iterative refinement must proceed until the resolution of the dyadic rationals is at least $R = \lceil \log_2 p \rceil$. This gives us the constant c:

$$1 = \sum_{\ell=0}^{p-1} a_\ell \phi_\ell^{\text{left}}(0) \quad \Rightarrow \quad c = \sum_{\ell=0}^{p-1} a_\ell z_\ell.$$

The four boundary scaling functions related to four vanishing moments are seen in fig. 2. There is a large resemblance between ϕ_3^{left} and the symmlet scaling function in fig. 1 (here denoted ϕ_4).

2.2 Right boundary scaling functions

Let again ϕ denote the interior symmlet Daubechies scaling function and p denote the number of vanishing moments of the associated wavelet. The idea for the right boundary scaling functions is the same as for the the left: We want a multiresolution analysis on $L^2((-\infty, 0])$ by modifiying the interior scaling functions. The ϕ_k with $\operatorname{supp} \phi_k \subset (-\infty, 0)$ are unaltered, but those with $\operatorname{supp} \phi_k \cap [0, \infty) \neq \emptyset$ are replaced by a corresponding ϕ_k^{right}. In conclusion, for $k = 0, \ldots, p-1$, the right boundary scaling functions satisfies the

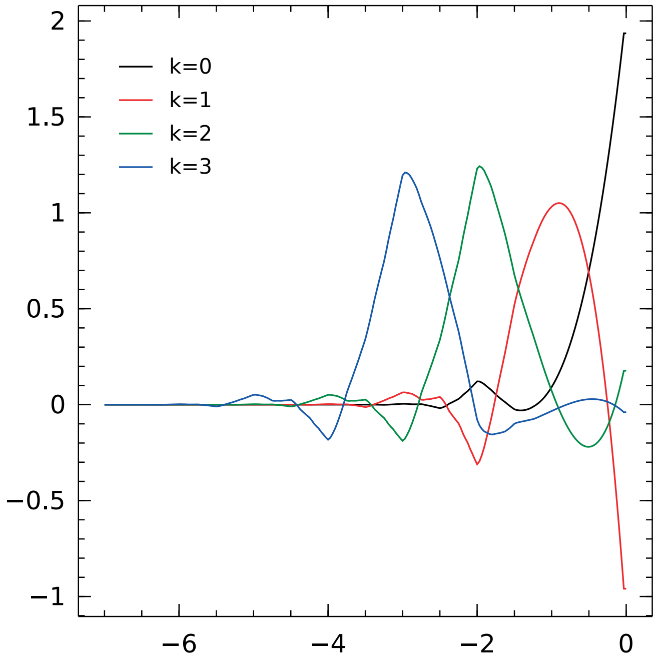

Figure 3: The right boundary scaling function with 4 vanishing moments.

dilation equations

$$\phi_k^{\text{right}}(x) = \sqrt{2}\sum_{l=0}^{p-1} H_{k,l}^{\text{right}}\phi_l^{\text{right}}(2x) + \sqrt{2}\sum_{m=p}^{p+2k} h_{k,m}^{\text{right}}\phi(2x+m+1), \tag{3}$$

where $(H_{k,l}^{\text{right}})$ and $(h_{k,m}^{\text{right}})$ are filter coefficients. The support of ϕ_k^{right} is $[-p-k, 0]$.

The four right boundary scaling functions related to four vanishing moments are seen in fig. 3.

3 Scaling functions on an interval

The left and right scaling functions are multiresolutions for the postive and negative halflines, respectively. We can combine the three kind of scaling functions to obtain a multiresoltion for an interval. Let us first recall a few properties of a multiresoltion analysis for $L^2(\mathbb{R})$. We use the notation

$$\phi_{J,k}(x) = 2^{J/2}\phi(2^J x - k)$$

and

$$V_J = \operatorname{span}\{\phi_{J,k} \mid k \in \mathbb{Z}\}$$

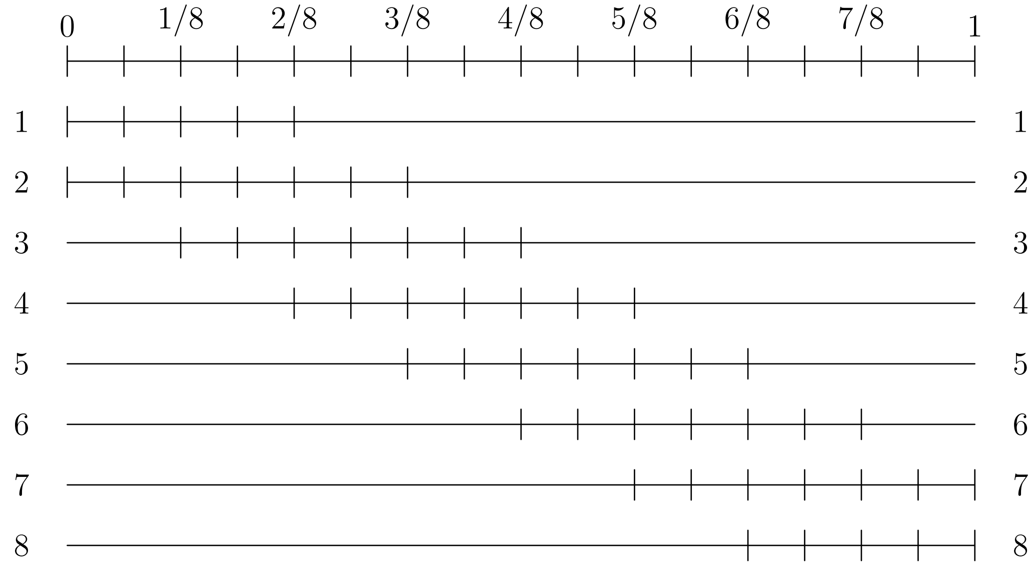

Figure 4: The support of the interval scaling functions with $p = 2$ vanishing moments at scale $J = 4$ computed at dyadic rationals of resolution $R = 4$.

Then

$$L^2(\mathbb{R}) = \overline{\bigcup_{J \in \mathbb{Z}} V_J}.$$

Initially we want a similar multiresoltion for the interval $[0, 1]$, that is, a relation of the form

$$L^2([0,1]) = \overline{\bigcup_{J \in \mathbb{Z}} \widetilde{V}_J}$$

for suitably defined $\widetilde{V}_J$. We index the basis functions in $\widetilde{V}_J$ as $\widetilde{\phi}_1, \ldots, \widetilde{\phi}_N$. The first thing to notice is that all three kinds of scaling functions must be dilated such that their support is at most of length 1. Just as in the classic multiresoltion we dilate with numbers of the form 2^J. Let τ_h deonte a translation operator, meaning that $(\tau_h f)(x) = f(x - h)$. Define

$$\widetilde{\phi}_k(x) = \begin{cases} \phi_k^{\text{left}}(x), & 1 \leq k \leq p, \\ \tau_k \phi(x) & p < k \leq N - p, \\ \tau_1 \phi_k^{\text{right}}(x) & N - p < k \leq N. \end{cases}$$

Then

$$\widetilde{V}_J = \operatorname{span}\{\widetilde{\phi}_{J,k} \mid 0 \leq k \leq N\}, \quad \widetilde{\phi}_{J,k}(x) = 2^{J/2} \widetilde{\phi}_k(2^J x).$$

As an example of the support of the interval basis functions consider fig. 4.

In *IntervalWavelets* we want to evalute the interval scaling functions at the dyadic rationals of a predefined resolution R. Note that we have an interaction between the scale and the resolution:

$$\widetilde{\phi}_{J,k}\left(\frac{n}{2^R}\right) = 2^{J/2}\widetilde{\phi}_k\left(2^J\frac{n}{2^R}\right) = 2^{J/2}\widetilde{\phi}_k\left(\frac{n}{2^{R-J}}\right).$$

This means that in order to evaluate $\widetilde{\phi}_{J,k}$ in dyadic rationals of resolution R we only need to compute $\widetilde{\phi}$ in dyadic rationals of resolution $R-J$.

References

[1] Albert Cohen, Ingrid Daubechies, and Pierre Vial. "Wavelets on the Interval and Fast Wavelet Transforms". In: *Applied and Computational Harmonic Analysis* 1.1 (Dec. 1993), pp. 54–81. DOI: 10.1006/acha.1993.1005.

[2] Stéphane Mallat. *A Wavelet Tour of Signal Processing. The Sparse Way*. 3rd ed. Academic Press, 2009.

[3] Gilbert Strang. "Wavelets and Dilation Equations: A Brief Introduction". In: *SIAM Review* 31.4 (1989), pp. 614–627. DOI: 10.1137/1031128.