Adding/Deleting Elements/Relations To/From a Poset

Let $P = (X, \leq)$ be a partially order set (or poset). Here X is a set of *elements* and $\leq$ is a reflexive, antisymmetric, transitive *relation* on X.

We wish to consider the following four operations: (1) add an element to a poset, (2) delete an element from a poset, (3) add a relation to a poset, and (4) delete a relation from a poset.

Adding/Deleting Elements

Operations on elements are relatively simple to describe. Let $P = (X, \leq)$ be a poset.

If $a \notin X$, then to add the element a to P results in a new poset that includes a in which a is related only to itself. Formally, let $P' = P + a$ be the poset $P' = (X', \leq')$ where we have the following:

(1) $X' = X \cup \{a\}$.

(2) $\leq' = \leq \cup \{(a, a)\}$. That is,

- $\forall x, y \in X, x \leq' y \iff x \leq y$,
- $\forall x \in X,\ x \not\leq' a$ and $a \not\leq' x$, and
- $a \leq' a$.

Element deletion is also easy to describe. Deleting an element a from P deletes a from the set X and all remaining elements have the same relations they had before. Formally, for $a \in X$, let $P' = P - a$ be the poset $P' = (X', \leq')$ where we have the following:

(1) $X' = X - \{a\}$.

(2) $\forall x, y \in X',\ x \leq' y \iff x \leq y$.

Note that element addition and deletion operations need not commute. While it is true that $(P + a) - a = P$, in general we have $(P - a) + a \neq P$.

Adding/Deleting Relations

Adding a relation to a poset requires us to include additional relations implied by transitivity. Let $P = (X, \leq)$ be a poset containing incomparable elements a and b.

We define $P+(a < b)$ to be the poset $P' = (X', \leq')$ in which we have the following:

- $X' = X$.
- $\forall x, y \in X',\ x \leq' y \iff (x \leq y)$ or $(x \leq a$ and $b \leq y)$.

Stated differently, $\leq'$ is the minimal superset of $\leq$ that includes the pair (a, b) and that is reflexive, antisymmetric, and transitive.

There does not appear to be "best" way to define relation deletion. Suppose $P = (X, \leq)$ is a poset in which $a < b$; we want to define $P' = P - (a < b)$. For example, suppose $P = ([3], \leq)$ is the total order $1 < 2 < 3$. How shall we define $P-(1 < 3)$? Since we delete $(1, 3)$ from the relation, we cannot have both $1 < 2$ and

$2 < 3$, so one of those must be deleted as well. This leads to two possible choices for $\leq'$ are these:

- $\leq' = \{(1,1),(2,2),(3,3),(1,2)\}$ and
- $\leq' = \{(1,1),(2,2),(3,3),(2,3)\}$.

There's no reasonable way to choose between these alternatives. Both are derived from $\leq$ with a minimum number of changes. So we take another approach by deleting both $1 < 2$ and $2 < 3$. This results in the antichain on [3].

More generally, when we delete $a < b$ from P we need to delete other relations. In particular, if there is an x with $a < x < b$, we cannot keep both $a < x$ and $x < b$. Our solution is to delete *both*.

Thus we define $P - (a < b)$ to be the poset $P' = (X', \leq')$ in which $X' = X$ and

$$\leq' = \leq - \{(a,b)\} - \{(a,x),(x,b) : a < x < b\}.$$

Claim. *P' is a poset.*

Proof. We need to check that $\leq'$ is reflexive, antisymmetric, and transitive.

Since we have not deleted any relation of the form (x,x) from $\leq$, it follows that $\leq'$ is reflexive.

Since $\leq' \subset \leq$ it follows that

$$(x \leq' y \text{ and } y \leq' x) \Rightarrow (x \leq y \text{ and } y \leq x) \Rightarrow x = y.$$

Finally, we must show that $\leq'$ is transitive. Suppose $x <' y <' z$ but we do not have $x <' z$. This means that (x,z) was a relation deleted from $\leq$ and so we have one of the following:

(1) $(x,z) = (a,b)$,
(2) $a = x < z < b$, or
(3) $a < x < z = b$.

Case (1) cannot hold because then we have $x = a < y < b = z$ in which case neither $x <' y$ nor $y <' z$ contradicting the supposition that $x <' y <' z$.

In case (2) we have that $a = x < y < z < b$ contradicting $x <' y$, and a similar contradiction holds in case (3). $\square$