Solving PDEs Associated with Economic Models

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This package EconPDEs.jl introduces a fast and robust way to solve systems of PDEs + algebraic equations (i.e. DAEs) associated with economic models. This note details the underlying algorithm.

Difference with Achdou et al. (2016) Achdou et al. (2016) focus on quasi-linear PDEs of the form

$$0 = f_1(V) + f_2(x)\partial_x V + f_3(x)\partial_{xx} V$$

In contrast, the package solves non-linear PDEs of the form

$$0 = f_1(V) + f_2(x, \partial_x V) + f_3(x, \partial_x V)\partial_{xx} V$$

Step 1: Write Finite Difference Scheme The system of PDEs is written on a state space grid and derivatives are substituted by finite difference approximations. As in Achdou et al. (2016), first order derivatives are upwinded. This allows to naturally handle boudnary counditions at the frontiers of the state space. This also tends to make the scheme monotonous.

Step 2: Solve Finite Difference Scheme Denote V the solution of the PDE and denote $F(V)$ the finite difference scheme corresponding to a model. The goal is to find V such that $F(V) = 0$. The package includes a solver especially written for these finite different schemes. This method is most similar to a method used in the fluid dynamics literature. In this context, it is called the Pseudo-Transient Continuation method, and is denoted Ψtc. Formal conditions for the convergence of the algorithm are given in Kelley and Keyes (1998).

To understand the intuition for the method, note that the existing literature in economics solves for V using using one of the two methods:

1. Non-linear solver. The method solves for the non-linear system $F(V) = 0$. A Newton-Raphson update takes the form

$$0 = F(y_t) + J_F(v_t)(v_{t+1} - v_t) \tag{1}$$

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The issue with this method is that it requires the initial guess to be sufficiently close to the solution.[1]

2. ODE solver . The method solves for the ODE $F(V) = \dot{V}$. The solution of $F(V) = 0$ is obtained with $T \to +\infty$. [2] With a simple explicit Euler method, updates take the form

$$0 = F(v_t) - \frac{1}{\Delta}(v_{t+1} - y_t) \tag{2}$$

This method tends to be slow, and does not always converge, depending on the ODE chosen.

I propose to solve for V using a fully implicit Euler method. Updates take the form

$$\forall t \leq T \qquad 0 = F(v_{t+1}) - \frac{1}{\Delta}(v_{t+1} - y_t)$$

Each time step now requires to solve a non-linear equation. I solve this non-linear equation using a Newton-Raphson method. These inner iterations therefore take the form

$$\forall i \leq I \qquad 0 = F(y_t^i) - \frac{1}{\Delta}(y_t^i - y_t) + (J_F(v_t^i) - \frac{1}{\Delta})(y_t^{i+1} - v_t^i) \tag{3}$$

We know that the Newton-Raphson method converges if the initial guess is close enough to the solution. Since y_t converges towards v_{t+1} as Δ tends to zero, one can always choose Δ low enough so that the inner steps converge.Therefore, I adjust Δ as follows. If the inner iterations do not converge, I decrease Δ. When the inner iteration converges, I increase Δ.

The update Equation (3) can be see as weighted average of the Newton-Raphson step Equation (1) and of the explicit Euler step Equation (2). After a few sucessful implicit time steps, Δ is large and therefore the algorithm becomes like Newton-Rapshon. In particular, the convergence is quadratic around the solution.

The algorithm with $I = 1$ and Δ constant corresponds to Achdou et al. (2016). Allowing $I > 1$ and adjusting Δ are important to ensure convergence in case on non-linear PDEs.

References

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[1] This method is usded, for instance, by Gârleanu and Panageas (2015)

[2] See, for instance, Di Tella (2016), Silva (2015).

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