

# 1 L'Hospital's Rule

Let's return to limits of the form  $f(x)/g(x)$  which have an indeterminate form of  $0/0$  if both are evaluated at  $c$ . The typical example being the limit considered by Euler:

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x}.$$

We know this is 1 using a bound from geometry, but might also guess this is one, as we know from linearization near 0 that we have

$$\sin(x) = x - \sin(\xi)x^2/2, \quad 0 < \xi < x.$$

This would yield:

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = \lim_{x \rightarrow 0} \frac{x - \sin(\xi)x^2/2}{x} = \lim_{x \rightarrow 0} 1 + \sin(\xi) \cdot x/2 = 1.$$

This is because we know  $\sin(\xi)x/2$  has a limit of 0, when  $|\xi| \leq |x|$ .

That doesn't look any easier, as we worried about the error term, but if just mentally replaced  $\sin(x)$  with  $x$  - which it basically is near 0 - then we can see that the limit should be the same as  $x/x$  which we know is 1 without thinking.

Basically, we found that in terms of limits, if both  $f(x)$  and  $g(x)$  are 0 at  $c$ , that we *might* be able to just take this limit:  $(f(c) + f'(c) \cdot (x - c))/(g(c) + g'(c) \cdot (x - c))$  which is just  $f'(c)/g'(c)$ .

Wouldn't that be nice? We could find difficult limits just by differentiating the top and the bottom at  $c$  (and not use the messy quotient rule).

Well, in fact that is more or less true, a fact that dates back to [L'Hospital](#) - who wrote the first textbook on differential calculus - though this result is likely due to one of the Bernoulli brothers.

**L'Hospital's rule:** Suppose,  $f$  and  $g$  are differentiable in  $(a, b)$  with  $a < c < b$ . Moreover, suppose  $|g(x)| > 0$  for all  $x$  in  $(a, b)$  except at  $c$ . Further suppose  $f(c) = g(c) = 0$ . If  $\lim_{x \rightarrow c} f'(x)/g'(x) = L$  (the limit exists), then  $L = \lim_{x \rightarrow c} f(x)/g(x)$ .

That is *if* the limit of  $f(x)/g(x)$  is indeterminate of the form  $0/0$ , but the limit of  $f'(x)/g'(x)$  is known, possibly by simple continuity, then the limit of  $f(x)/g(x)$  exists and is equal to that of  $f'(x)/g'(x)$ .

To apply this rule to Euler's example,  $\sin(x)/x$ , we just need to consider that:

$$L = 1 = \lim_{x \rightarrow 0} \frac{\cos(x)}{1},$$

So, as well,  $\lim_{x \rightarrow 0} \sin(x)/x = 1$ .

This is due to  $\cos(x)$  being continuous at 0, so this limit is just  $\cos(0)/1$ . (More importantly, the tangent line expansion of  $\sin(x)$  at 0 is  $\sin(0) + \cos(0)x$ , so that  $\cos(0)$  is why this answer is as it is, but we don't need to think in terms of  $\cos(0)$ , but rather the tangent-line expansion, which is  $\sin(x) \approx x$ , as  $\cos(0)$  appears as the coefficient.

## Examples

- Consider this limit at 0:  $(a^x - 1)/x$ . We have  $f(x) = a^x - 1$  has  $f(0) = 0$ , so this limit is indeterminate of the form  $0/0$ . The derivative of  $f(x)$  is  $f'(x) = a^x \log(a)$  which has  $f'(0) = \log(a)$ . The derivative of the bottom is also 1 at 0, so we have:

$$\log(a) = \frac{\log(a)}{1} = \frac{f'(0)}{g'(0)} = \lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} \frac{a^x - 1}{x}.$$

(Why rewrite in the "opposite" direction? Because the theorem's result ( $L$  is the limit) is only true if the related limit involving the derivative exists.)

- Consider this limit:

$$\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{x}.$$

It too is of the indeterminate form  $0/0$ . The derivative of the top is  $e^x + e^{-x}$ , which is 2 when  $x = 0$ , so the ratio of  $f'(0)/g'(0)$  is seen to be 2. By L'Hospital's rule, the limit above is 2.

- Sometimes, L'Hospital's rule must be applied twice. Consider this limit:

$$\lim_{x \rightarrow 0} \frac{\cos(x)}{1 - x^2}$$

By L'Hospital's rule *if* this following limit exists, the two will be equal:

$$\lim_{x \rightarrow 0} \frac{-\sin(x)}{-2x}.$$

But if we didn't guess the answer, we see that this new problem is *also* indeterminate of the form  $0/0$ . So, repeating the process, this new limit will exist and be equal to the following limit, should it exist:

$$\lim_{x \rightarrow 0} \frac{-\cos(x)}{-2} = 1/2.$$

As  $L = 1/2$  for this related limit, it must also be the limit of the original problem, by L'Hospital's rule.

- Our "intuitive" limits can bump into issues. Take for example the limit of  $(\sin(x) - x)/x^2$  as  $x$  goes to 0. Using  $\sin(x) \approx x$  makes this look like  $0/x^2$  which is still indeterminate. (Because the difference is higher order than  $x$ .) Using L'Hospital's, says this limit will exist (and be equal) if the following one does:

$$\lim_{x \rightarrow 0} \frac{\cos(x) - 1}{2x}.$$

This particular limit is indeterminate of the form  $0/0$ , so we again try L'Hospital's rule and consider

$$\lim_{x \rightarrow 0} \frac{-\sin(x)}{2} = 0$$

So as this limit exists, working backwards, the original limit in question will also be 0.

- This example comes from the Wikipedia page. It "proves" a discrete approximation for the second derivative.

Show if  $f''(x)$  exists at  $c$  and is continuous at  $c$ , then

$$f''(c) = \lim_{h \rightarrow 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2}.$$

This will follow from two applications of L'Hospital's rule to the right-hand side. The first says, the limit on the right is equal to this limit, should it exist:

$$\lim_{h \rightarrow 0} \frac{f'(c+h) - 0 - f'(c-h)}{2h}.$$

We have to be careful, as we differentiate in the  $h$  variable, not the  $c$  one, so the chain rule brings out the minus sign. But again, as we still have an indeterminate form  $0/0$ , this limit will equal the following limit should it exist:

$$\lim_{h \rightarrow 0} \frac{f''(c+h) - 0 - (-f''(c-h))}{2} = \lim_{c \rightarrow 0} \frac{f''(c+h) + f''(c-h)}{2} = f''(c).$$

That last equality follows, as it is assumed that  $f''(x)$  exists at  $c$  and is continuous, that is,  $f''(c \pm h) \rightarrow f''(c)$ .

The expression above finds use when second derivatives are numerically approximated. (The middle expression is the basis of the central-finite difference approximation to the derivative.)

## 1.1 Why?

The proof of L'Hospital's rule takes advantage of Cauchy's [generalization](#) of the mean value theorem to two functions. Let  $f(x)$  and  $g(x)$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$  so that on  $[c, c+x]$  there exists a  $\xi$  with  $f'(\xi) \cdot (f(x) - f(c)) = g'(\xi) \cdot (g(x) - g(c))$ . In our formulation, both  $f(c)$  and  $g(c)$  are zero, so we have, provided we know that  $g(x)$  is non zero, that  $f(x)/g(x) = f'(\xi)/g'(\xi)$  for some  $\xi$ ,  $c < \xi < c+x$ . That the right-hand side has a limit as  $x \rightarrow c+$  is true by the assumption that the limit of the derivative's ratio exists. (The  $\xi$  part can be removed by considering it as a composition of a function going to  $c$ .) Thus the right limit of the left hand side is known. Similarly, working with  $[c-x, c]$  we can get the left limit is known and is equal to the right.

### 1.1.1 L'Hospital's picture

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## 1.2 Generalizations

L'Hospital's rule generalizes to other indeterminate forms, in particular  $\infty/\infty$  can be proved at the same time as  $0/0$  with a more careful [proof](#).

In addition, indeterminate forms of the type  $0 \cdot \infty$ ,  $0^0$  and  $\infty^\infty$  can be re-expressed to be in the form  $0/0$  or  $\infty/\infty$ .

For example, consider

$$\lim_{x \rightarrow \infty} \frac{x}{e^x}.$$

We see it is of the form  $\infty/\infty$  (That we are taking a limit at  $\infty$  is also a generalization.) We have by the generalized L'Hospital rule that this limit will exist and be equal to this one, should it exist:

$$\lim_{x \rightarrow \infty} \frac{1}{e^x}.$$

This limit is, of course, 0, as it is of the form  $1/\infty$ . It is not hard to build up from here to show that for any integer value of  $n > 0$  that:

$$\lim_{x \rightarrow \infty} \frac{x^n}{e^x} = 0.$$

This is an expression of the fact that exponential functions grow faster than polynomial functions.

## Examples

- What is the limit  $x \log(x)$  as  $x \rightarrow 0+$ ?

Rewriting, we see this is just:

$$\lim_{x \rightarrow 0+} \frac{\log(x)}{1/x}.$$

L'Hospital's rule clearly applies to one sided limits, as well as two (our proof sketch used one-sided limits), so this limit will equal the following, should it exist:

$$\lim_{x \rightarrow 0+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0+} -x = 0.$$

- What is the limit  $x^x$  as  $x \rightarrow 0+$ ? The expression is of the form  $0^0$ , which is indeterminate. (Even though floating point math defines the value as 1.) We can rewrite this by taking a log:

$$x^x = \exp(\log(x^x)) = \exp(x \log(x)) = \exp(\log(x)/(1/x)).$$

Be just saw that  $\lim_{x \rightarrow 0+} \log(x)/(1/x) = 0$ . So by the rules for limits of compositions and the fact that  $e^x$  is continuous, we see  $\lim_{x \rightarrow 0+} x^x = e^0 = 1$ .

- L'Hospital himself was interested in this limit for  $a > 0$

$$\lim_{x \rightarrow a} \frac{\sqrt{2a^3 \cdot x - x^4} - a \cdot (a^2 \cdot x)^{1/3}}{a - (a \cdot x^3)^{1/4}}.$$

These derivatives can be done by hand, but to avoid any minor mistakes we utilize SymPy taking care to use rational numbers for the fractional powers, so as not to lose precision through floating point roundoff:

```

using CalculusWithJulia # loads `SymPy`
@vars a x positive=true real=true
f(x) = sqrt(2a^3*x - x^4) - a * (a^2*x)^(1//3)
g(x) = a - (a*x^3)^(1//4)

```

```

g (generic function with 1 method)

```

We can see that at  $x = a$  we have the indeterminate form  $0/0$ :

```

f(a), g(a)

```

```

(0, 0)

```

What about the derivatives?

```

fp, gp = subs(diff(f(x),x), x=>a), subs(diff(g(x),x), x=>a)

```

```

(-4*a/3, -3/4)

```

Their ratio will not be indeterminate, so the limit in question is just the ratio:

```

fp/gp

```

$$\frac{16a}{9}$$

Of course, we could have just relied on `limit`, which knows about L'Hospital's rule:

```

limit(f(x)/g(x), x, a)

```

$$\frac{16a}{9}$$

### 1.3 Questions

⊗ Question

This function  $f(x) = \sin(5x)/x$  is *indeterminate* at  $x = 0$ . What type?

1.  $0/0$

2.  $\infty/\infty$

3.

$$0^0$$

4.

$$\infty - \infty$$

5.

$$0 \cdot \infty$$

⊗ Question

This function  $f(x) = \sin(x)^{\sin(x)}$  is *indeterminate* at  $x = 0$ . What type?

1.

$$0/0$$

2.

$$\infty/\infty$$

3.

$$0^0$$

4.

$$\infty - \infty$$

5.

$$0 \cdot \infty$$

⊗ Question

This function  $f(x) = (x - 2)/(x^2 - 4)$  is *indeterminate* at  $x = 2$ . What type?

1.

$$0/0$$

2.

$$\infty/\infty$$

3.

$$0^0$$

4.

$$\infty - \infty$$

5.

$$0 \cdot \infty$$

⊗ Question

This function  $f(x) = (g(x + h) - g(x - h))/(2h)$  ( $g$  is continuous) is *indeterminate* at  $h = 0$ . What type?

1.

$$0/0$$

2.

$$\infty/\infty$$

3.

$$0^0$$

4.

$$\infty - \infty$$

5.

$$0 \cdot \infty$$

⊗ Question

This function  $f(x) = x \log(x)$  is *indeterminate* at  $x = 0$ . What type?

1.

$$0/0$$

2.

$$\infty/\infty$$

3.

$$0^0$$

4.

$$\infty - \infty$$

5.

$$0 \cdot \infty$$

⊗ Question

Does L'Hospital's rule apply to this limit:

$$\lim_{x \rightarrow \pi} \frac{\sin(\pi x)}{\pi x}.$$

1. No. It is not indeterminate

2. Yes. It is of the form  $0/0$

⊗ Question

Use L'Hospital's rule to find the limit

$$L = \lim_{x \rightarrow 0} \frac{4x - \sin(x)}{x}.$$

What is  $L$ ?

⊗ Question

Use L'Hospital's rule to find the limit

$$L = \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x}.$$

What is  $L$ ?

⊗ Question

Use L'Hospital's rule *two* or more times to find the limit

$$L = \lim_{x \rightarrow 0} \frac{x - \sin(x)}{x^3}.$$

What is  $L$ ?

⊗ Question

Use L'Hospital's rule *two* or more times to find the limit

$$L = \lim_{x \rightarrow 0} \frac{1 - x^2/2 - \cos(x)}{x^3}.$$

What is  $L$ ?

⊗ Question

By using a common denominator to rewrite this expression, use L'Hospital's rule to find the limit

$$L = \lim_{x \rightarrow 0} \frac{1}{x} - \frac{1}{\sin(x)}.$$

What is  $L$ ?

**Question** Use L'Hospital's rule to find the limit

$$L = \lim_{x \rightarrow \infty} \log(x)/x$$

What is  $L$ ?

**Question** Using L'Hospital's rule, does

$$\lim_{x \rightarrow 0^+} x^{\log(x)}$$

exist?

Consider  $x^{\log(x)} = e^{\log(x) \log(x)}$ .

1. Yes
2. No

**Question** Using L'Hospital's rule, find the limit of

$$\lim_{x \rightarrow 1} (2 - x)^{\tan(\pi/2 \cdot x)}.$$

(Hint, express as  $\exp^{\tan(\pi/2 \cdot x) \cdot \log(2-x)}$  and take the limit of the resulting exponent.)

1. It does not exist

2.

0

3.

$2\pi$

4.

$e^{2/\pi}$

5.

1