1 The Gradient, Divergence, and Curl

The gradient of a scalar function $f: R^n \rightarrow R$ is a vector field of partial derivatives. In R^2, we have:

$$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle.$$

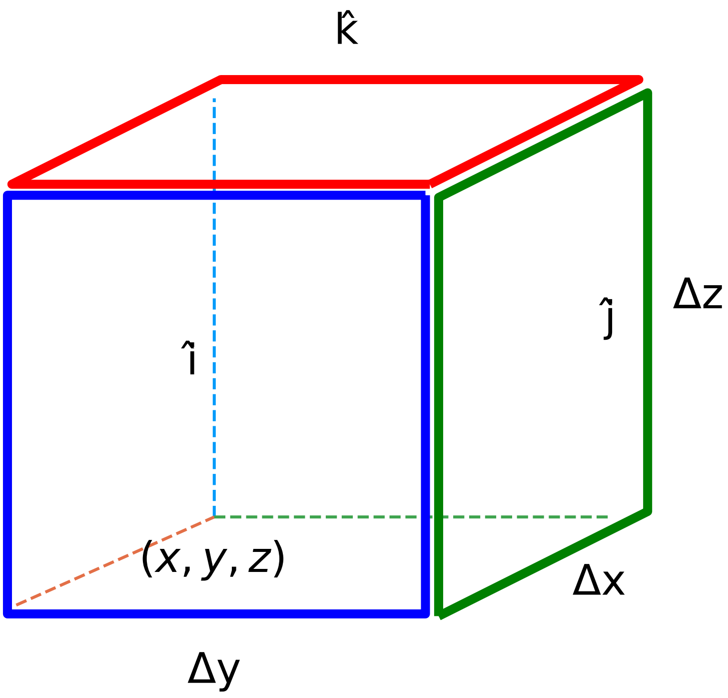
It has the interpretation of pointing out the direction of greatest ascent for the surface $z = f(x, y)$.

We move now to two other operations, the divergence and the curl, which combine to give a language to describe vector fields in R^3.

```
using CalculusWithJulia
using Plots
```

1.1 The divergence

Let $F: R^3 \rightarrow R^3 = \langle F_x, F_y, F_z \rangle$ be a vector field. Consider now a small box-like region, R, with surface, S, on the cartesian grid, with sides of length Δx, Δy, and Δz with (x, y, z) being one corner. The outward pointing unit normals are $\pm\hat{i}, \pm\hat{j}$, and $\pm\hat{k}$.



Consider the sides with outward normal $\hat{i}$. The contribution to the surface integral, $\oint_S (F \cdot \hat{N}) dS$, could be *approximated* by

$$\left(F(x + \Delta x, y, z) \cdot \hat{i}\right) \Delta y \Delta z,$$

whereas, the contribution for the face with outward normal $-\hat{\imath}$ could be approximated by:

$$\left(F(x,y,z) \cdot (-\hat{\imath})\right) \Delta y \Delta z.$$

The functions are being evaluated at a point on the face of the surface. For Riemann integrable functions, any point in a partition may be chosen, so our choice will not restrict the generality.

The total contribution of the two would be:

$$\left(F(x+\Delta x,y,z) \cdot \hat{\imath}\right) \Delta y \Delta z + \left(F(x,y,z) \cdot (-\hat{\imath})\right) \Delta y \Delta z = \left(F_x(x+\Delta x,y,z) - F_x(x,y,z)\right) \Delta y \Delta z,$$

as $F \cdot \hat{\imath} = F_x$.

Were we to divide by $\Delta V = \Delta x \Delta y \Delta z$ *and* take a limit as the volume shrinks, the limit would be $\partial F/\partial x$.

If this is repeated for the other two pair of matching faces, we get a definition for the *divergence*:

> The *divergence* of a vector field $F : R^3 \rightarrow R^3$ is given by $\text{divergence}(F) = \lim \frac{1}{\Delta V} \oint_S F \cdot \hat{N} dS = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$.

The limit expression for the divergence will hold for any smooth closed surface, S, converging on (x,y,z), not just box-like ones.

1.1.1 General n

The derivation of the divergence is done for $n=3$, but could also have easily been done for two dimensions ($n=2$) or higher dimensions $n>3$. The formula in general would be: for $F(x_1, x_2, \dots, x_n) : R^n \rightarrow R^n$:

$$\text{divergence}(F) = \sum_{i=1}^n \frac{\partial F_i}{\partial x_i}.$$

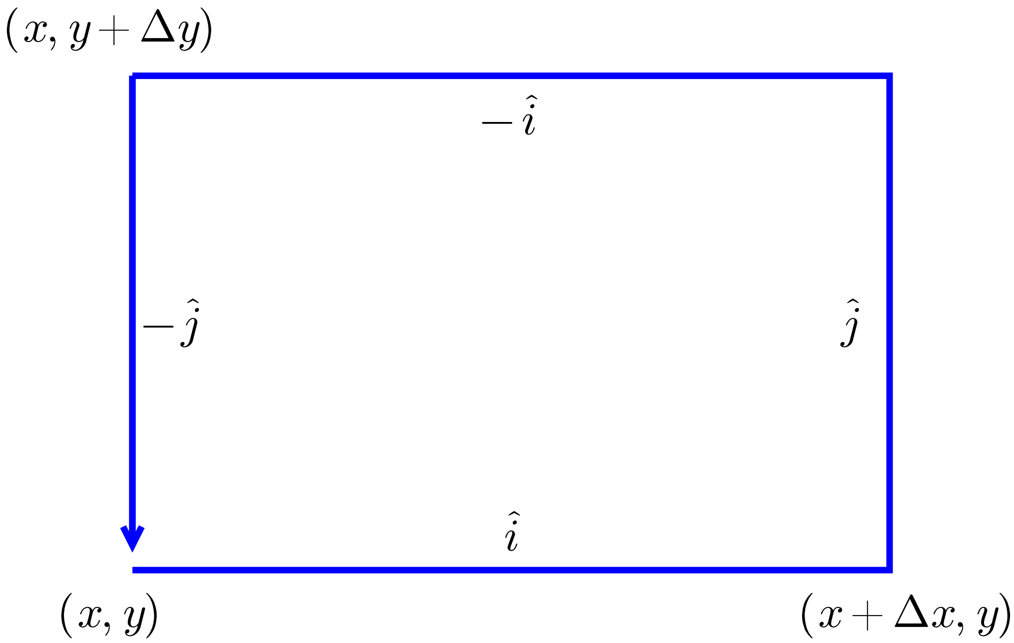
In `Julia`, the divergence can be implemented different ways depending on how the problem is presented. Here are two functions from the `CalculusWithJulia` package for when the problem is symbolic or numeric:

```
divergence(F::Vector{Sym}, vars) = sum(diff.(F, vars))
divergence(F::Function, pt) = sum(diag(ForwardDiff.jacobian(F, pt)))
```

The latter being a bit inefficient, as all n^2 partial derivatives are found, but only the n diagonal ones are used.

1.2 The curl

Before considering the curl for $n=3$, we derive a related quantity in $n=2$. The "curl" will be a measure of the microscopic circulation of a vector field. To that end we consider a microscopic box-region in R^2:



Let $F = \langle F_x, F_y \rangle$. For small enough values of Δx and Δy the line integral, $\oint_C F \cdot d\vec{r}$ can be *approximated* by 4 terms:

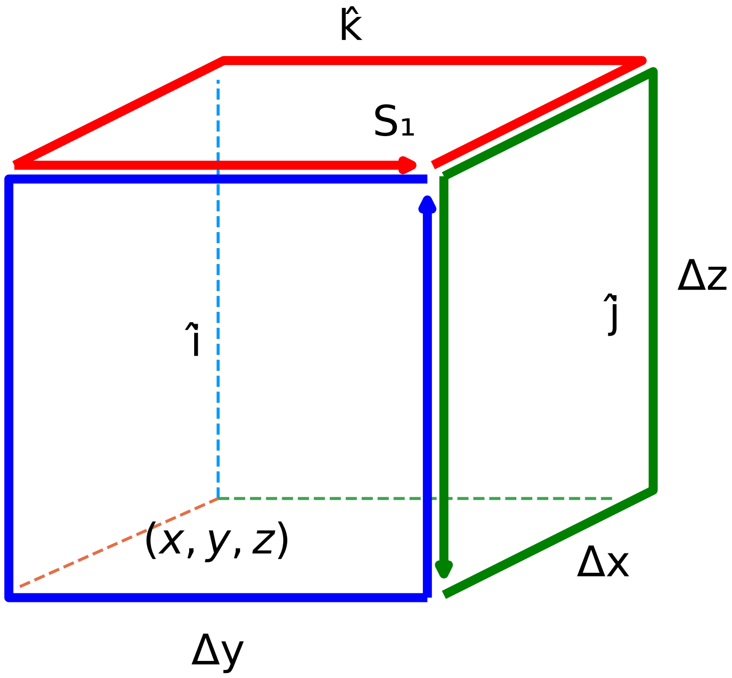
$$
\begin{aligned}
&\left(F(x,y)\cdot\hat{i}\right)\Delta x + \left(F(x+\Delta x,y)\cdot\hat{j}\right)\Delta y + \left(F(x,y+\Delta y)\cdot(-\hat{i})\right)\Delta x + \left(F(x,y)\cdot(-\hat{j})\right)\Delta x && (1)\\
&= F_x(x,y)\Delta x + F_y(x+\Delta x,y)\Delta y + F_x(x,y+\Delta y)(-\Delta x) + F_y(x,y)(-\Delta y) && (2)\\
&= (F_y(x+\Delta x,y) - F_y(x,y))\Delta y - (F_x(x,y+\Delta y) - F_x(x,y))\Delta x. && (3)
\end{aligned}
$$

The Riemann approximation allows a choice of evaluation point for Riemann integrable functions, and the choice here lends itself to further analysis. Were the above divided by $\Delta x \Delta y$, the area of the box, and a limit taken, partial derivatives appear to suggest this formula:

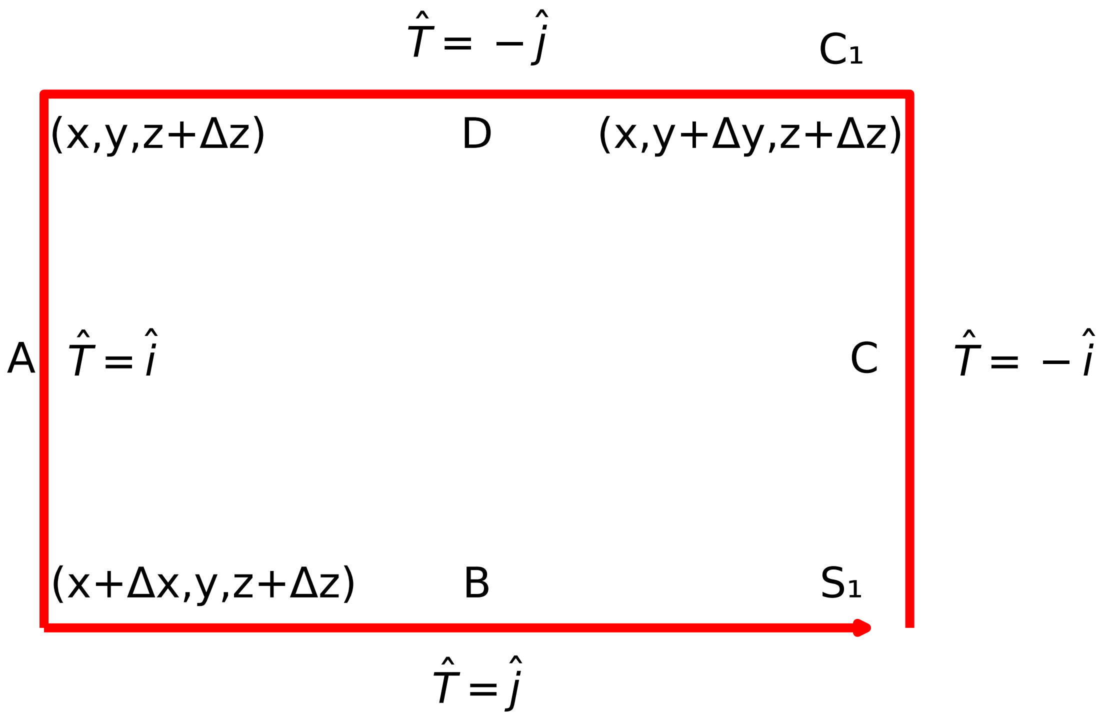
$$\lim \frac{1}{\Delta x \Delta y} \oint_C F \cdot d\vec{r} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}.$$

The scalar function on the right hand side is called the (two-dimensional) curl of F and the left-hand side lends itself as a measure of the microscopic circulation of the vector field, $F : R^2 \to R^2$.

Consider now a similar scenario for the $n = 3$ case. Let $F = \langle F_x, F_y, F_z \rangle$ be a vector field and S a box-like region with side lengths Δx, Δy, and Δz, anchored at (x, y, z).



The box-like volume in space with the top area, with normal $\hat{k}$, designated as S_1. The curve C_1 traces around S_1 in a counter clockwise manner, consistent with the right-hand rule pointing in the outward normal direction. The face S_1 with unit normal $\hat{k}$ looks like:



Now we compute the *line integral*. Consider the top face, S_1, connecting $(x, y, z + \Delta z), (x +$

$\Delta x, y, z + \Delta z), (x + \Delta x, y + \Delta y, z + \Delta z), (x, y + \Delta y, z + \Delta z)$, Using the *right hand rule*, parameterize the boundary curve, C_1, in a counter clockwise direction so the right hand rule yields the outward pointing normal ($\hat{k}$). Then the integral $\oint_{C_1} F \cdot \hat{T} ds$ is *approximated* by the following Riemann sum of 4 terms:

$$F(x,y,z+\Delta z)\cdot\hat{\imath}\Delta x + F(x+\Delta x, y, z+\Delta z)\cdot\hat{\jmath}\Delta y + F(x, y+\Delta y, z+\Delta z)\cdot(-\hat{\imath})\Delta x + F(x,y,z+\Delta z)\cdot(-\hat{\jmath})\Delta y.$$

(The points c_i are chosen from the endpoints of the line segments.)

$$\oint_{C_1} F\cdot\hat{T}ds \approx (F_y(x+\Delta x, y, z+\Delta z) - F_y(x,y,z+\Delta z))\Delta y - (F_x(x, y+\Delta y, z+\Delta z) - F_x(x,y,z+\Delta z))\Delta x$$

As before, were this divided by the *area* of the surface, we have after rearranging and cancellation:

$$\frac{1}{\Delta S_1}\oint_{C_1} F\cdot\hat{T}ds \approx \frac{F_y(x+\Delta x, y, z+\Delta z) - F_y(x,y,z+\Delta z)}{\Delta x} - \frac{F_x(x, y+\Delta y, z+\Delta z) - F_x(x,y,z+\Delta z)}{\Delta y}.$$

In the limit, as $\Delta S \rightarrow 0$, this will converge to $\partial F_y/\partial x - \partial F_x/\partial y$.

Had the bottom of the box been used, a similar result would be found, up to a minus sign.

Unlike the two dimensional case, there are other directions to consider and here the other sides will yield different answers. Consider now the face connecting $(x, y, z), (x+\Delta x, y, z), (x+\Delta x, y, z + \Delta z)$, and $ (x,y,z+\Delta{z})$ with outward pointing normal $-\hat{\jmath}$. Let S_2 denote this face and C_2 describe its boundary. Orient this curve so that the right hand rule points in the $-\hat{\jmath}$ direction (the outward pointing normal). Then, as before, we can approximate:

$$\oint_{C_2} F \cdot \hat{T} ds \approx F(x,y,z) \cdot \hat{\imath}\Delta x + F(x+\Delta x, y, z) \cdot \hat{k}\Delta z + F(x,y,z+\Delta z) \cdot (-\hat{\imath})\Delta x + F(x,y,z) \cdot (-\hat{k})\Delta z \tag{4}$$

$$= (F_z(x+\Delta x, y, z) - F_z(x,y,z))\Delta z - (F_x(x,y,z+\Delta z) - F(x,y,z))\Delta x. \tag{5}$$

Dividing by $\Delta S = \Delta x \Delta z$ and taking a limit will give:

$$\lim \frac{1}{\Delta S}\oint_{C_2} F \cdot \hat{T} ds = \frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}.$$

Had, the opposite face with outward normal $\hat{\jmath}$ been chosen, the answer would differ by a factor of -1.

Similarly, let S_3 be the face with outward normal $\hat{\imath}$ and curve C_3 bounding it with parameterization chosen so that the right hand rule points in the direction of $\hat{\imath}$. This will give

$$\lim \frac{1}{\Delta S}\oint_{C_3} F \cdot \hat{T} ds = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}.$$

In short, depending on the face chosen, a different answer is given, but all have the same type.