1 Average of a function and the mean value theorem for integrals

Let $f(x)$ be a continuous function over the interval $[a, b]$ with $a < b$.

The average value of f over $[a, b]$ is defined by:

$$\frac{1}{b-a} \int_a^b f(x) dx.$$

If f is a constant, this is just the contant value, as would be expected. If f is *piecewise* linear, then this is the weighted average of these constants.

1.1 Examples

Average velocity The average velocity between times $a < b$, is simply the change in position during the time interval divided by the change in time. In notation, this would be $(x(b) - x(a))/(b - a)$. If $v(t) = x'(t)$ is the velocity, then by the second part of the fundamental theorem of calculus, we have, in agreement with the definition above, that:

$$\text{average velocity} = \frac{x(b) - x(a)}{b - a} = \frac{1}{b-a} \int_a^b v(t) dt.$$

The average speed is the change in *total* distance over time, which is given by

$$\text{average speed} = \frac{1}{b-a} \int_a^b |v(t)| dt.$$

Let $\bar{v}$ be the average velocity. Then we have $\bar{v} \cdot (b-a) = x(b) - x(a)$, or the change in position can be written as a constant $(\bar{v})$ times the time, as though we had a constant velocity. This is an old intuition. Bressoud comments on the special case known to scholars at Merton College around 1350 that the distance traveled by an object under uniformly increasing velocity starting at v_0 and ending at v_t is equal to the distance traveled by an object with constant velocity of $(v_0 + v_t)/2$.

Example What is the average value of $f(x) = \sin(x)$ over $[0, \pi]$?

$$\text{average} = \frac{1}{\pi - 0} \int_0^\pi \sin(x) dx = \frac{1}{\pi} (-\cos(x)) \Big|_0^\pi = \frac{2}{\pi}$$

Visually, we have:

```
using CalculusWithJulia
using Plots
plot(sin, 0, pi)
plot!(x -> 2/pi, 0, 2pi)
```

```
Plot{Plots.PlotlyBackend() n=2}
```

Example What is the average value of the function f which is 1 between $[0,3]$, 2 between $(3,5]$ and 1 between $(5,6]$?

Though not continuous, $f(x)$ is integrable as it contains only jumps. The integral from $[0,6]$ can be computed with geometry: $3 \cdot 3 + 2 \cdot 2 + 1 \cdot 1 = 14$. The average then is $14/(6-0) = 7/3$.

Example What is the average value of the function e^{-x} between 0 and $\log(2)$?

$$\text{average} = \frac{1}{\log(2) - 0} \int_0^{\log(2)} e^{-x} dx = \frac{1}{\log(2)} (-e^{-x}) \Big|_0^{\log(2)} = -\frac{1}{\log(2)} (\frac{1}{2} - 1) = \frac{1}{2\log(2)}.$$

Visualizing, we have

```
plot(x -> exp(-x), 0, log(2))
plot!(x -> 1/(2*log(2)), 0, log(2))
```

```
Plot{Plots.PlotlyBackend() n=2}
```

1.2 The mean value theorem for integrals

If $f(x)$ is assumed integrable, the average value of $f(x)$ is defined, as above. Re-expressing gives that there exists a K with

$$K \cdot (b - a) = \int_a^b f(x) dx.$$

When we assume that $f(x)$ is continuous, we can describe K as a value in the range of f:

> **The mean value theorem for integrals**: Let $f(x)$ be a continuous function on $[a,b]$ with $a < b$. Then there exists c with $a \leq c \leq b$ with $f(c) \cdot (b-a) = \int_a^b f(x) dx$.

The proof comes from the intermediate value theorem and the extreme value theorem. Since f is continuous on a closed interval, there exists values m and M with $f(c_m) = m \leq f(x) \leq M = f(c_M)$, for some c_m and c_M in the interval $[a,b]$. Since $m \leq f(x) \leq M$, we must have:

$$m \cdot (b-a) \leq K \cdot (b-a) \leq M \cdot (b-a).$$

So in particular K is in $[m, M]$. But m and M correspond to values of $f(x)$, so by the intermediate value theorem, $K = f(c)$ for some c that must lie in between c_m and c_M, which means as well that it must be in $[a,b]$.

Proof of second part of Fundamental Theorem of Calculus The mean value theorem is exactly what is needed to prove formally the second part of the Fundamental Theorem of Calculus. Again, suppose $f(x)$ is continuous on $[a,b]$ with $a < b$. For any $a < x < b$, we define $F(x) = \int_a^x f(u) du$. Then the derivative of F is f.

Let $h > 0$. Then consider the forward difference $(F(x+h) - F(x))/h$. Rewriting gives:

$$\frac{\int_a^{x+h} f(u)du - \int_a^x f(u)du}{h} = \frac{\int_x^{x+h} f(u)du}{h} = f(\xi(h)).$$

The value $\xi(h)$ is just the c corresponding to a given value in $[x, x+h]$ guaranteed by the mean value theorem. We only know that $x \leq \xi(h) \leq x+h$. But this is plenty - it says that $\lim_{h\rightarrow 0+} \xi(h) = x$. Using the fact that f is continuous and the known properties of limits of compositions of functions this gives $\lim_{h\rightarrow 0+} f(\xi(h)) = f(x)$. But this means that the (right) limit of the secant line expression exists and is equal to $f(x)$, which is what we want to prove. Repeating a similar argument when $h < 0$, finishes the proof.

The basic notion used is simply that for small h, this expression is well approximated by the left Riemann sum taken over $[x, x+h]$:

$$f(\xi(h)) \cdot h = \int_x^{x+h} f(u)du.$$

1.3 Questions

⊛ Question
Between 0 and 1 a function is constantly 1. Between 1 and 2 the function is constantly 2. What is the average value of the function over the interval $[0, 2]$?

⊛ Question
Between 0 and 2 a function is constantly 1. Between 2 and 3 the function is constantly 2. What is the average value of the function over the interval $[0, 3]$?

⊛ Question
What integral will show the intuition of the Merton College scholars that the distance traveled by an object under uniformly increasing velocity starting at v_0 and ending at v_t is equal to the distance traveled by an object with constant velocity of $(v_0 + v_t)/2$.

1. $$\int_0^t (v(0) + v(u))/2du = v(0)/2 \cdot t + x(u)/2 \Big|_0^t$$

2. $$(v(0) + v(t))/2 \cdot \int_0^t du = (v(0) + v(t))/2 \cdot t$$

3. $$\int_0^t v(u)du = v^2/2\Big|_0^t$$

⊛ Question
Find the average value of $\cos(x)$ over the interval $[-\pi/2, \pi/2]$.

⊛ Question
Find the average value of $\cos(x)$ over the interval $[0, \pi]$.

⊛ Question
Find the average value of $f(x) = e^{-2x}$ between 0 and 2.

⊛ Question
Find the average value of $f(x) = \sin(x)^2$ over the 0, π.

⊛ Question
Which is bigger? The average value of $f(x) = x^{10}$ or the average value of $g(x) = |x|$ over the interval $[0, 1]$?

1. That of $f(x) = x^{10}$.
2. That of $g(x) = |x|$.

⊛ Question
Define a family of functions over the interval $[0, 1]$ by $f(x; a, b) = x^a \cdot (1 - x)^b$. Which has a greater average, $f(x; 2, 3)$ or $f(x; 3, 4)$?

1. $$f(x; 3, 4)$$
2. $$f(x; 2, 3)$$

⊛ Question
Suppose the average value of $f(x)$ over $[a, b]$ is 100. What is the average value of $100f(x)$ over $[a, b]$?

⊛ Question
Suppose $f(x)$ is continuous and positive on $[a, b]$.

- Explain why for any $x > a$ it must be that:

$$F(x) = \int_a^x f(x)dx > 0$$

1. Because the definite integral is only defined for positive area, so it is always positive
2. Because the mean value theorem says this is $f(c)(x - a)$ for some c and both terms are positive by the assumptions

- Explain why $F(x)$ is increasing.

1. By the fundamental theorem of calculus, part I, $F'(x) = f(x) > 0$, hence $F(x)$ is increasing
2. By the intermediate value theorem, as $F(x) > 0$, it must be true that $F(x)$ is increasing
3. By the extreme value theorem, $F(x)$ must reach its maximum, hence it must increase.

⊛ Question
For $f(x) = x^2$, which is bigger: the average of the function $f(x)$ over $[0, 1]$ or the geometric mean which is the exponential of the average of the logarithm of f over the same interval?

1. The average of f
2. The exponential of the average of $\log(f)$