1 Fundamental Theorem or Calculus

We refer to the example the section on transformations where two operators on functions were defined:

$$D(f)(k) = f(k) - f(k-1), \quad S(f)(k) = f(1) + f(2) + \cdots + f(k).$$

It was remarked that these relationships hold: $D(S(f))(k) = f(k)$ and $S(D(f))(k) = f(k) - f(0)$. These being a consequence of the inverse relationship between addition and subtraction. These two relationships are examples of a more general pair of relationships known as the Fundamental Theorem of Calculus or FTC.

The FTC details the interconnectivity between the operations of integration and differentiation.

For example:

> What is the definite integral of the derivative? That is, what is $A = \int_a^b f'(x)dx$? (Assume f' is continuous.)

To investigate, we begin with the right Riemann sum using $h = (b-a)/n$:

$$A \approx S_n = \sum_{i=1}^{n} f'(a+ih) \cdot h.$$

But the Mean Value Theorem says that for small h we have $f'(x) \approx (f(x) - f(x-h))/h$. Using this approximation with $x = a + ih$ gives:

$$A \approx \sum_{i=1}^{n} (f(a+ih) - f(a+(i-1)h)).$$

If we let $g(i) = f(a+ih)$, then the summand above is just $g(i) - g(i-1) = D(g)(i)$ and the above then is just the sum of the $D(g)(i)$s, or:

$$A \approx S(D(g))(n) = g(n) - g(0).$$

But $g(n) - g(0) = f(a+nh) - f(a+0h) = f(b) - f(a)$. That is we expect that if $\approx$ in the limit becomes $=$ then:

$$\int_a^b f'(x)dx = f(b) - f(a).$$

This is indeed the case.

The other question would be

> What is the derivative of the integral? That is, can we find the derivative of $\int_0^x f(u)du$?

Let's look first at the integral using the right-Riemann sum, again using $h = (b-a)/n$:

$$\int_a^b f(u)du \approx f(a+1h)h + f(a+2h)h + \cdots f(a+nh)h = S(g)(n),$$

where we define $g(i) = f(a+ih)h$. In the above, n relates to b, but we could have stopped accumulating at any value. The analog for $S(g)(k)$ would be $\int_a^x f(u)du$ where $x = a+kh$. That is we can make a function out of integration by considering the mapping $(x, \int_a^x f(u)du)$. This might be written as $F(x) = \int_a^x f(u)du$. With this definition, can we take a derivative in x?

Again, we fix a large n and let $h = (b-a)/n$. And suppose $x = a+Mh$ for some M. Then writing out the approximations to both the definite integral and derivative we have

$$\begin{aligned} F'(x) &= \frac{d}{dx}\int_a^x f(u)du \\ &\approx \frac{F(x)-F(x-h)}{h} = \frac{\int_a^x f(u)du - \int_a^{x-h} f(u)du}{h} \\ &\approx \frac{(f(a+1h)h + f(a+2h)h + \cdots + f(a+(M-1)h)h + f(a+Mh)h)}{h} - \frac{(f(a+1h)h + f(a+2}{} \end{aligned}$$

If $g(i) = f(a+ih)$, then the above becomes

$$\begin{aligned} F'(x) &\approx D(S(g))(M) \\ &= f(a+Mh) \\ &= f(x). \end{aligned}$$

That is $F'(x) \approx f(x)$.

In the limit, then, we would expect that

$$\frac{d}{dx}\int_a^x f(u)du = f(x).$$

With these heuristics, we now have:

The Fundamental Theorem of Calculus

Part 1: Let f be a continuous function on a closed interval $[a,b]$ and define $F(x) = \int_a^x f(u)du$ for $a \leq x \leq b$. Then F is continuous on $[a,b]$, differentiable on (a,b) and moreover, $F'(x) = f(x)$.

Part 2: Now suppose f is any integrable function on a closed interval $[a,b]$ and $F(x)$ is *any* differentiable function on $[a,b]$ with $F'(x) = f(x)$. Then $\int_a^b f(x)dx = F(b) - F(a)$.

In Part 1, the integral $F(x) = \int_a^x f(u)du$ is defined for any Riemann integrable function, f. If the function is not continuous, then it is true the F will be continuous, but it need not be true that it is differentiable at all points in (a,b). Forming F from f is a form of *smoothing*. It makes a continuous function out of an integrable one, a differentiable function from a continuous one, and a $k+1$-times differentiable function from a k-times differentiable one.

1.1 Using the Fundamental Theorem of Calculus to evaluate definite integrals

The major use of the FTC is the computation of $\int_a^b f(x)dx$. Rather then resort to Riemann sums or geometric arguments, there is an alternative - find a function F with $F'(x) = f(x)$ and compute $F(b) - F(a)$.

Some examples:

- Consider the problem of Archimedes, $\int_0^1 x^2 dx$. Clearly, we have with $f(x) = x^2$ that $F(x) = x^3/3$ will satisfy the assumptions of the FTC, so that:

$$\int_0^1 x^2 dx = F(1) - F(0) = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}.$$

- More generally, we know if $n \neq -1$ that if $f(x) = x^n$, that

$$F(x) = x^{n+1}/(n+1)$$

will satisfy $F'(x) = f(x)$, so that

$$\int_a^b x^n dx = \frac{b^{n+1} - a^{n+1}}{n+1}, \quad n \neq -1.$$

(Well almost! We must be careful to know that $a \cdot b > 0$, as otherwise we will encounter a place where $f(x)$ may not be integrable.)

We note that the above includes the case of a constant, or $n = 0$: $\int 1dx = x^1/1 = x$.

What about the case $n = -1$, or $f(x) = 1/x$, that is not covered by the above? For this special case, it is known that $F(x) = \log(x)$ (natural log) will have $F'(x) = 1/x$. This gives for $0 < a < b$:

$$\int_a^b \frac{1}{x} dx = \log(b) - \log(a).$$

- Let $f(x) = \cos(x)$. How much area is between $-\pi/2$ and $\pi/2$? We have that $F(x) = \sin(x)$ will have $F'(x) = f(x)$, so:

$$\int_{-\pi/2}^{\pi/2} \cos(x) dx = F(\pi/2) - F(-\pi/2) = 1 - (-1) = 2.$$

1.1.1 An alternate notation for $F(b) - F(a)$

The expression $F(b) - F(a)$ is often written in this more compact form:

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big|_{x=a}^b, \text{ or just expr}\Big|_{x=a}^b.$$

The vertical bar is used for the *evaluation* step, in this case the a and b mirror that of the definite integral. This notation lends itself to working inline, as we illustrate with this next problem where we "know" a function "F", so just express it "inline":

$$\int_0^{\pi/4} \sec^2(x)dx = \tan(x)\Big|_{x=0}^{\pi/4} = 1 - 0 = 1.$$

A consequence of this notation is:

$$F(x)\Big|_{x=a}^{b} = -F(x)\Big|_{x=b}^{a}.$$

This says nothing more than $F(b) - F(a) = -F(a) - (-F(b))$, though more compactly.

1.2 The indefinite integral

A function $F(x)$ with $F'(x) = f(x)$ is known as an *antiderivative* of f. For a given f, there are infinitely many antiderivatives: if $F(x)$ is one, then so is $G(x) = F(x) + C$. But - due to the mean value theorem - all antiderivatives for f differ at most by a constant.

The **indefinite integral** of $f(x)$ is denoted by:

$$\int f(x)dx.$$

(There are no limits of integration.) There are two possible definitions: this refers to the set of *all* antiderivatives, or is just one of the set of all antiderivatives for f. The former gives rise to expressions such as

$$\int x^2 dx = \frac{x^3}{3} + C$$

where C is the *constant of integration* and isn't really a fixed constant, but any possible constant. These notes will follow the lead of `SymPy` and not give a C in the expression, but instead rely on the reader to understand that there could be many other possible expressions given, though all differ by no more than a constant. This means, that $\int f(x)dx$ refers to *an* antiderivative, not *the* antiderivative.

`SymPy` provides the `integrate` function to perform integration. There are two usages:

- `integrate(ex, var)` to find an indefinite integral. For a univariate function `f` this is shortened to `integrate(f)`,
- `integrate(ex, (var, a, b))` to find the definite integral. This integrates the expression in the variable `var` from `a` to `b`. For a univariate function `f` this can also be shortened to `integrate(f, a, b)`, which matches our usual template of `action(function, args...)`.

To illustrate, we have, this call finds an antiderivative:

```
using CalculusWithJulia  # loads `SymPy`, `QuadGK`
using Plots
integrate(sin)
```

$$-\cos(x)$$

Whereas this call computes the "area" under $f(x)$ between `a` and `b`:

```
integrate(sin, 0, pi)
```

$$2.0$$

In the last two example, function objects were integrated. We can also integrate expressions, though in this usage we must specify the variable:

```
@vars x n real=true
integrate(x^n, x)          # indefinite integral
```

$$\begin{cases} \frac{x^{n+1}}{n+1} & \text{for } n \neq -1 \\ \log(x) & \text{otherwise} \end{cases}$$

For a definite integral we have

```
integrate(acos(1-x), (x, 0, 2))
```

$$\pi$$

1.3 Rules of integration

The `integrate` function includes an implementation of the Risch algorithm. This algorithm is implemented for elementary functions and operations involving these functions, such as addition, multiplication, division and composition. There are some "rules" of integration that allow this to work.

- The integral of a constant times a function:

$$\int c \cdot f(x)dx = c \cdot \int f(x)dx.$$

This follows as if $F(x)$ is an antiderivative of $f(x)$, then $[cF(x)]' = cf(x)$ by rules of derivatives.

- The integral of a sum of functions:

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx.$$

This follows immediately as if $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$, then $[F(x) + G(x)]' = f(x) + g(x)$, so the right hand side will have a derivative of $f(x) + g(x)$.

In fact, this more general form where c and d are constants covers both cases:

$$\int (cf(x) + dg(x))dx = c\int f(x)dx + d\int g(x)dx.$$

This statement is nothing more than the derivative formula $[cf(x)+dg(x)]' = cf'(x)+dg'(x)$. The product rule gives rise to a technique called *integration by parts* and the chain rule gives rise to a technique of *integration by substitution*, but we defer those discussions to other sections.

Examples

- The antiderivative of the polynomial $p(x) = a_n x^n + \cdots a_1 x + a_0$ follows from the linearity of the integral and the general power rule:

$$\begin{aligned}
\int (a_n x^n + \cdots a_1 x + a_0)dx &= \int a_n x^n dx + \cdots \int a_1 x dx + \int a_0 dx && (1)\\
&= a_n \int x^n dx + \cdots + a_1 \int x dx + a_0 \int dx && (2)\\
&= a_n \frac{x^{n1}}{n+1} + \cdots + a_1 \frac{x^2}{2} + a_0 \frac{x}{1}. && (3)
\end{aligned}$$

- More generally, a Laurent polynomial allows for terms with negative powers. These too can be handled by the above. For example

$$\begin{aligned}
\int (\frac{2}{x} + 2 + 2x)dx &= \int \frac{2}{x}dx + \int 2dx + \int 2xdx && (4)\\
&= 2\int \frac{1}{x}dx + 2\int dx + 2\int xdx && (5)\\
&= 2\log(x) + 2x + 2\frac{x^2}{2}. && (6)
\end{aligned}$$

- Consider this integral:

$$\int_0^\pi 100\sin(x)dx = F(\pi) - F(0),$$

where $F(x)$ is an antiderivative of $100\sin(x)$. But:

$$\int 100\sin(x)dx = 100\int \sin(x)dx = 100(-\cos(x)).$$

So the answer to the question is

$$\int_0^\pi 100\sin(x)dx = (100(-\cos(\pi))) - (100(-\cos(0))) = (100(-(-1))) - (100(-1)) = 200.$$

This seems like a lot of work, and indeed it is more than is needed. The following would be more typical once the rules are learned:

$$\int_0^\pi 100\sin(x)dx = -100(-\cos(x))\Big|_0^\pi = 100\cos(x)\Big|_\pi^0 = 100(1) - 100(-1) = 200.$$

1.4 The derivative of the integral

The relationship that $[\int_a^x f(u)du]' = f(x)$ is a bit harder to appreciate, as it doesn't help answer many ready made questions. Here we give some examples of its use.

First, the expression defining an antiderivative, or indefinite integral, is given in term of a definite integral:

$$F(x) = \int_a^x f(u)du.$$

The value of a does not matter, as long as the integral is defined.

XXX can not include '.gif' file here

The picture for this, for non-negative f, is of accumulating area as x increases. It can be used to give insight into some formulas:

For any function, we know that $F(b) - F(c) + F(c) - F(a) = F(b) - F(a)$. For this specific function, this translates into this property of the integral:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Similarly, $\int_a^a f(x)dx = F(a) - F(a) = 0$ follows.

To see that the value of a does not matter, consider $a_0 < a_1$. Then we have with

$$F(x) = \int_{a_0}^x f(u)du, \quad G(x) = \int_{a_0}^x f(u)du,$$

That $F(x) = G(x) + \int_{a_0}^{a_1} f(u)du$. The additional part may look complicated, but the point is that as far as x is involved, it is a constant. Hence both F and G are antiderivatives if either one is.

Example In probability theory, for a positive, continuous random variable, the probability that the random value is less than a is given by $P(X \leq a) = F(a) = \int_0^a f(x)dx$. (Positive means the integral starts at 0, whereas in general it could be $-\infty$, a minor complication that we haven't yet discussed.)

For example, the exponential distribution with rate 1 has $f(x) = e^{-x}$. Compute $F(x)$.

This is just $F(x) = \int_0^x e^{-u}du = -e^{-u}\Big|_0^x = 1 - e^{-x}$.

The "uniform" distribution on $[a, b]$ has

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases}$$

Find $f(x)$. There are some subtleties here. If we assume that $F(x) = \int_0^x f(u)du$ then we know if $f(x)$ is continuous that $F'(x) = f(x)$. Differentiating we get

$$f(x) = \begin{cases} 0 & x < a \\ \frac{1}{b-a} & a < x < b \\ 0 & x > b \end{cases}$$

However, the function f is *not* continuous on $[a, b]$ and $F'(x)$ is not differentiable on (a, b). It is true that f is integrable, and where F is differentiable $F' = f$. So f is determined except possibly at the points $x = a$ and $x = b$.

Example The error function is defined by $\text{erf}(x) = 2/\sqrt{\pi} \int_0^x e^{-u^2} du$. It is implemented in `Julia` through `erf`. Suppose, we were to ask where it takes on it's maximum value, what would we find?

The answer will either be at a critical point, at 0 or as x goes to ∞. We can differentiate to find critical points:

$$[\text{erf}(x)] = \frac{2}{\pi} e^{-x^2}.$$

Oh, this is never 0, so there are no critical points. The maximum occurs at 0 or as x goes to ∞. Clearly at 0, we have $\text{erf}(0) = 0$, so the answer will be as x goes to ∞.

In retrospect, this is a silly question. As $f(x) > 0$ for all x, we *must* have that $F(x)$ is strictly increasing, so never gets to a local maximum.

Example The Dawson function is

$$F(x) = e^{-x^2} \int_0^x e^{t^2} dt$$

Characterize any local maxima or minima.

For this we need to consider the product rule. The fundamental theorem of calculus will help with the right-hand side. We have:

$$F'(x) = (-2x)e^{-x^2} \int_0^x e^{t^2} dt + e^{-x^2} e^{x^2} = -2xF(x) + 1$$

We need to figure out when this is 0. For that, we use some numeric math.

```
F(x) = exp(-x^2) * quadgk(t -> exp(t^2), 0, x)[1]
Fp(x) = -2x*F(x) + 1
cps = find_zeros(Fp, -4,4)
```

```
2-element Array{Float64,1}:
 -0.9241388730045916
  0.9241388730045916
```

We could take a second derivative to characterize. For that we use $F''(x) = [-2xF(x)+1]' = -2F(x) + -2x(-2xF(x) + 1)$, so

```
Fpp(x) = -2F(x) + 4x^2*F(x) - 2x
Fpp.(cps)
```

```
2-element Array{Float64,1}:
  1.0820884492703637
 -1.0820884492703637
```

The first value being positive says there is a relative minimum at -0.924139, at 0.924139 there is a relative maximum.

Example From the familiar formula rate $\times$ time $=$ distance, we "know," for example, that a car traveling 60 miles an hour for one hour will have traveled 60 miles. This allows us to translate statements about the speed (or more generally velocity) into statements about position at a given time. If the speed is not constant, we don't have such an easy conversion.

Suppose our velocity at time t is $v(t)$, and always positive. We want to find the position at time t, $x(t)$. Let's assume $x(0) = 0$. Let h be some small time step, say $h = (t-0)/n$ for some large $n > 0$. Then we can *approximate* $v(t)$ between $[ih, (i+1)h)$ by $v(ih)$. This is a constant so the change in position over the time interval $[ih, (i+1)h)$ would simply be $v(ih) \cdot h$, and ignoring the accumulated errors, the approximate position at time t would be found by adding this pieces together: $x(t) \approx v(0h) \cdot h + v(1h) \cdot h + v(2h) \cdot h + \cdots + v(nh)h$. But we recognize this (as did Beeckman in 1618) as nothing more than an approximation for the Riemann sum of v over the interval $[0, t]$. That is, we expect:

$$x(t) = \int_0^t v(u)du.$$

Hopefully this makes sense: our position is the result of accumulating our change in position over small units of time. The old one-foot-in-front-of-another approach to walking out the door.

The above was simplified by the assumption that $x(0) = 0$. What if $x(0) = x_0$ for some non-zero value. Then the above is not exactly correct, as $\int_0^0 v(u)du = 0$. So instead, we might write this more concretely as:

$$x(t) = x_0 + \int_0^t v(u)du.$$

There is a similar relationship between velocity and acceleration, but let's think about it formally. If we know that the acceleration is the rate of change of velocity, then we have $a(t) = v'(t)$. By the FTC, then

$$\int_0^t a(u)du = \int_0^t v'(t) = v(t) - v(0).$$

Rewriting gives a similar statement as before:

$$v(t) = v_0 + \int_0^t a(u)du.$$

Example A junior engineer at `Treadmills.com` is tasked with updating the display of calories burned for an older-model treadmill. The old display involved a sequence of LED ”dots” that updated each minute. The last 10 minutes were displayed. Each dot corresponded to one calorie burned, so the total number of calories burned in the past 10 minutes was the number of dots displayed, or the sum of each column of dots. An example might be:

```
  **
  ****
  *****
 ********
**********
```

In this example display there was 1 calorie burned in the first minute, then 2, then 5, 5, 4, 3, 2, 2, 1. The total is 24.

In her work the junior engineer found this old function for updating the display

```
function cnew = update(Cnew, Cold)
  cnew = Cnew - Cold
end
```

She discovered that the function was written awhile ago, and in MATLAB. The function receives the values `Cnew` and `Cold` which indicate the *total* number of calories burned up until that time frame. The value `cnew` is the number of calories burned in the minute. (Some other engineer has cleverly figured out how many calories have been burned during the time on the machine.)

The new display will have twice as many dots, so the display can be updated every 30 seconds and still display 10 minutes worth of data. What should the `update` function now look like?

Her first attempt was simply to rewrite the function in `Julia`:

```
function update(Cnew, Cold)
  cnew = Cnew - Cold
end
```

```
update (generic function with 1 method)
```

This has the advantage that each ”dot” still represents a calorie burned, so that a user can still count the dots to see the total burned in the past 10 minutes.

```
     * *
    ****** *
 ************* *
```

Sadly though, users didn't like it. Instead of a set of dots being, say, 5 high, they were now 3 high and 2 high. It "looked" like they were doing less work! What to do?

The users actually were not responding to the number of dots, which hadn't changed, but rather the *area* that they represented - and this shrank in half. (It is much easier to visualize area than count dots when tired.) How to adjust for that?

Well our engineer knew - double the dots and count each as half a calorie. This makes the "area" constant. She also generalized letting `n` be the number of updates per minute, in anticipation of even further improvements in the display technology:

```
function update(Cnew, Cold, n)
   cnew = (Cnew - Cold) * n
end
```

```
update (generic function with 2 methods)
```

Then the "area" represented by the dots stays fixed over this time frame.

The engineer then thought a bit more, as the form of her answer seemed familiar. She decides to parameterize it in terms of t and found with $h = 1/n$: `c(t) = (C(t) - C(t-h))/h`. Ahh - the derivative approximation. But then what is the "area"? It is no longer just the sum of the dots, but in terms of the functions she finds that each column represents $c(t) \cdot h$, and the sum is just $c(t_1)h + c(t_2)h + \cdots c(t_n)h$ which looks like an approximate integral.

If the display were to reach the modern age and replace LED "dots" with a higher-pixel display, then the function to display would be $c(t) = C'(t)$ and the area displayed would be $\int_{t-10}^{t} c(u)du$.

Thinking a bit harder, she knows that her `update` function is getting $C(t)$, and displaying the *rate* of calorie burn leads to the area displayed being interpretable as the total calories burned between t and $t-10$ (or $C(t) - C(t-10)$) by the Fundamental Theorem of Calculus.

1.5 Questions

⊛ Question
If $F(x) = e^{x^2}$ is an antiderivative for f, find $\int_0^2 f(x)dx$.

⊛ Question
If $\sin(x) - x\cos(x)$ is an antiderivative for $x\sin(x)$, find the following integral $\int_0^\pi x\sin(x)dx$.

⊛ Question
Find an antiderivative then evaluate $\int_0^1 x(1-x)dx$.

⊛ Question
Use the fact that $[e^x]' = e^x$ to evaluate $\int_0^e (e^x - 1)dx$.

⊛ Question
Find the value of $\int_0^1 (1 - x^2/2 + x^4/24)dx$.

⊛ Question
Using `SymPy`, what is an antiderivative for $x^2\sin(x)$?

1.
$$-x^2 \cos(x)$$

2.
$$-x^2 \cos(x) + 2x \sin(x)$$

3.
$$-x^2 \cos(x) + 2x \sin(x) + 2\cos(x)$$

⊛ Question
Using `SymPy`, what is an antiderivative for xe^{-x}?

1.
$$-e^{-x}$$

2.
$$-xe^{-x}$$

3.
$$-(1+x)e^{-x}$$

4.
$$-(1+x+x^2)e^{-x}$$

⊛ Question
Using `SymPy`, integrate the function $\int_0^{2\pi} e^x \cdot \sin(x) dx$.

⊛ Question
A particle has velocity $v(t) = 2t^2 - t$ between 0 and 1. If $x(0) = 0$, find the position $x(1)$.

⊛ Question
A particle has acceleration given by $\sin(t)$ between 0 and π. If the initial velocity is $v(0) = 0$, find $v(\pi/2)$.

⊛ Question
The position of a particle is given by $x(t) = \int_0^t g(u)du$, where $x(0) = 0$ and $g(u)$ is given by this piecewise linear graph:

```
Plot{Plots.PlotlyBackend() n=1}
```

- The velocity of the particle is positive over:

1. It is always positive
2. It is always negative
3. Between 0 and 1
4. Between 1 and 5

- The position of the particle is 0 at $t = 0$ and:

1.
$$t = 1$$

2.
$$t = 2$$

3.
$$t = 3$$

4.
$$t = 4$$

- The position of the particle at time $t = 5$ is?

- On the interval $[2, 3]$:

1. The position, $x(t)$, stays constant
2. The position, $x(t)$, increases with a slope of 1
3. The position, $x(t)$, increases quadratically from $-1/2$ to 1
4. The position, $x(t)$, increases quadratically from 0 to 1

⊛ Question
Let $F(x) = \int_{t-10}^{t} f(u)du$ for $f(u)$ a positive, continuous function. What is $F'(t)$?

1.
$$f(t)$$

2.
$$-f(t-10)$$

3.
$$f(t) - f(t-10)$$

⊛ Question
Suppose $f(x) \geq 0$ and $F(x) = \int_0^x f(u)du$. $F(x)$ is continuous and so has a maximum value on the interval $[0, 1]$ taken at some c in $[0, 1]$. It is

1. At a critical point
2. At the endpoint 0
3. At the endpoint 1

⊛ Question
Suppose $f(x)$ is monotonically decreasing with $f(0) = 1$, $f(1/2) = 0$ and $f(1) = -1$. Let $F(x) = \int_0^x f(u)du$. $F(x)$ is continuous and so has a maximum value on the interval $[0, 1]$ taken at some c in $[0, 1]$. It is

1. At a critical point, either 0 or 1
2. At a critical point, 1/2
3. At the endpoint 0
4. At the endpoint 1

⊛ Question
Barrow presented a version of the Fundamental Theorem of Calculus in a 1670 volume edited by Newton, Barrow's student (cf. Wagner). His version can be stated as follows (cf. Jardine):

Consider the following figure where f is a strictly increasing function with $f(0) = 0.$ and $x > 0$. The function $A(x) = \int_0^x f(u)du$ is also plotted. The point Q is $f(x)$, and the point P is $A(x)$. The point T is chosen to so that the length between T and x times the length between Q and x equals the length from P to x. ($|Tx| \cdot |Qx| = |Px|$.) Barrow showed that the line segment PT is tangent to the graph of $A(x)$. This figure illustrates the labeling for some function:

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Plot{Plots.PlotlyBackend() n=5}
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The fact that $|Tx| \cdot |Qx| = |Px|$ says what in terms of $f(x)$, $A(x)$ and $A'(x)$?

1. $$|Tx| \cdot f(x) = A(x)$$
2. $$A(x)/|Tx| = A'(x)$$
3. $$A(x) \cdot A'(x) = f(x)$$

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Solving, we get:

1. $$A(x) = A^2(x)/f(x)$$
2. $$A'(x) = A(x)$$

3.
$$A'(x) = f(x)$$

4.
$$A(x) = f(x)$$

⊛ Question
According to Bressoud "Newton observes that the rate of change of an accumulated quantity is the rate at which that quantity is accumulating". Which part of the FTC does this refer to:

1. Part 1: $[\int_a^x f(u)du]' = f$
2. Part 2: $\int_a^b f(u)du = F(b) - F(a)$.