1 L'Hospital's Rule

Let's return to limits of the form $f(x)/g(x)$ which have an indeterminate form of $0/0$ if both are evaluated at c. The typical example being the limit considered by Euler:

$$\lim_{x \to 0} \frac{\sin(x)}{x}.$$

We know this is 1 using a bound from geometry, but might also guess this is one, as we know from linearization near 0 that we have

$$\sin(x) = x - \sin(\xi)x^2/2, \quad 0 < \xi < x.$$

This would yield:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{x - \sin(\xi)x^2/2}{x} = \lim_{x \to 0} 1 + \sin(\xi) \cdot x/2 = 1.$$

This is because we know $\sin(\xi)x/2$ has a limit of 0, when $|\xi| \leq |x|$.

That doesn't look any easier, as we worried about the error term, but if just mentally replaced $\sin(x)$ with x - which it basically is near 0 - then we can see that the limit should be the same as x/x which we know is 1 without thinking.

Basically, we found that in terms of limits, if both $f(x)$ and $g(x)$ are 0 at c, that we *might* be able to just take this limit: $(f(c) + f'(c) \cdot (x - c))/(g(c) + g'(c) \cdot (x - c))$ which is just $f'(c)/g'(c)$.

Wouldn't that be nice? We could find difficult limits just by differentiating the top and the bottom at c (and not use the messy quotient rule).

Well, in fact that is more or less true, a fact that dates back to L'Hospital - who wrote the first textbook on differential calculus - though this result is likely due to one of the Bernoulli brothers.

> **L'Hospital's rule**: Suppose, f and g are differentiable in (a, b) with $a < c < b$. Moreover, suppose $|g(x)| > 0$ for all x in (a, b) except at c. Further suppose $f(c) = g(c) = 0$. If $\lim_{x \to c} f'(x)/g'(x) = L$ (the limit exists), then $L = \lim_{x \to c} f(x)/g(x)$.

That is *if* the limit of $f(x)/g(x)$ is indeterminate of the form $0/0$, but the limit of $f'(x)/g'(x)$ is known, possibly by simple continuity, then the limit of $f(x)/g(x)$ exists and is equal to that of $f'(x)/g'(x)$.

To apply this rule to Euler's example, $\sin(x)/x$, we just need to consider that:

$$L = 1 = \lim_{x \to 0} \frac{\cos(x)}{1},$$

So, as well, $\lim_{x \to 0} \sin(x)/x = 1$.

This is due to $\cos(x)$ being continuous at 0, so this limit is just $\cos(0)/1$. (More importantly, the tangent line expansion of $\sin(x)$ at 0 is $\sin(0) + \cos(0)x$, so that $\cos(0)$ is why this answer is as it is, but we don't need to think in terms of $\cos(0)$, but rather the tangent-line expansion, which is $\sin(x) \approx x$, as $\cos(0)$ appears as the coefficient.

Examples

- Consider this limit at 0: $(a^x - 1)/x$. We have $f(x) = a^x - 1$ has $f(0) = 0$, so this limit is indeterminate of the form $0/0$. The derivative of $f(x)$ is $f'(x) = a^x \log(a)$ which has $f'(0) = \log(a)$. The derivative of the bottom is also 1 at 0, so we have:

$$\log(a) = \frac{\log(a)}{1} = \frac{f'(0)}{g'(0)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{a^x - 1}{x}.$$

(Why rewrite in the "opposite" direction? Because the theorem's result (L is the limit) is only true if the related limit involving the derivative exists.)

- Consider this limit:

$$\lim_{x \to 0} \frac{e^x - e^{-x}}{x}.$$

It too is of the indeterminate form $0/0$. The derivative of the top is $e^x + e^{-x}$, which is 2 when $x = 0$, so the ratio of $f'(0)/g'(0)$ is seen to be 2. By L'Hospital's rule, the limit above is 2.

- Sometimes, L'Hospital's rule must be applied twice. Consider this limit:

$$\lim_{x \to 0} \frac{\cos(x)}{1 - x^2}$$

By L'Hospital's rule *if* this following limit exists, the two will be equal:

$$\lim_{x \to 0} \frac{-\sin(x)}{-2x}.$$

But if we didn't guess the answer, we see that this new problem is *also* indeterminate of the form $0/0$. So, repeating the process, this new limit will exist and be equal to the following limit, should it exist:

$$\lim_{x \to 0} \frac{-\cos(x)}{-2} = 1/2.$$

As $L = 1/2$ for this related limit, it must also be the limit of the original problem, by L'Hospital's rule.

- Our "intuitive" limits can bump into issues. Take for example the limit of $(\sin(x) - x)/x^2$ as x goes to 0. Using $\sin(x) \approx x$ makes this look like $0/x^2$ which is still indeterminate. (Because the difference is higher order than x.) Using L'Hospitals, says this limit will exist (and be equal) if the following one does:

$$\lim_{x \to 0} \frac{\cos(x) - 1}{2x}.$$

This particular limit is indeterminate of the form $0/0$, so we again try L'Hospital's rule and consider

$$\lim_{x \to 0} \frac{-\sin(x)}{2} = 0$$

So as this limit exists, working backwards, the original limit in question will also be 0.

- This example comes from the Wikipedia page. It ”proves” a discrete approximation for the second derivative.

Show if $f''(x)$ exists at c and is continuous at c, then

$$f''(c) = \lim_{h\to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2}.$$

This will follow from two applications of L'Hospital's rule to the right-hand side. The first says, the limit on the right is equal to this limit, should it exist:

$$\lim_{h\to 0} \frac{f'(c+h) - 0 - f'(c-h)}{2h}.$$

We have to be careful, as we differentiate in the h variable, not the c one, so the chain rule brings out the minus sign. But again, as we still have an indeterminate form $0/0$, this limit will equal the following limit should it exist:

$$\lim_{h\to 0} \frac{f''(c+h) - 0 - (-f''(c-h))}{2} = \lim_{c\to 0} \frac{f''(c+h) + f''(c-h)}{2} = f''(c).$$

That last equality follows, as it is assumed that $f''(x)$ exists at c and is continuous, that is, $f''(c \pm h) \to f''(c)$.

The expression above finds use when second derivatives are numerically approximated. (The middle expression is the basis of the central-finite difference approximation to the derivative.)

1.1 Why?

The proof of L'Hospital's rule takes advantage of Cauchy's generalization of the mean value theorem to two functions. Let $f(x)$ and $g(x)$ be continuous on $[a, b]$ and differentiable on (a, b) so that on $[c, c+x]$ there exists a ξ with $f'(\xi) \cdot (f(x) - f(c)) = g'(\xi) \cdot (g(x) - g(c))$. In our formulation, both $f(c)$ and $g(c)$ are zero, so we have, provided we know that $g(x)$ is non zero, that $f(x)/g(x) = f'(\xi)/g'(\xi)$ for some ξ, $c < \xi < c + x$. That the right-hand side has a limit as $x \to c+$ is true by the assumption that the limit of the derivative's ratio exists. (The ξ part can be removed by considering it as a composition of a function going to c.) Thus the right limit of the left hand side is known. Similarly, working with $[c - x, c]$ we can get the left limit is known and is equal to the right.

1.1.1 L'Hospital's picture

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1.2 Generalizations

L'Hospital's rule generalizes to other indeterminate forms, in particular ∞/∞ can be proved at the same time as $0/0$ with a more careful proof.

In addition, indeterminate forms of the type $0 \cdot \infty$, 0^0 and ∞^∞ can be re-expressed to be in the form $0/0$ or ∞/∞.

For example, consider

$$\lim_{x\to\infty} \frac{x}{e^x}.$$

We see it is of the form ∞/∞ (That we are taking a limit at ∞ is also a generalization.) We have by the generalized L'Hospital rule that this limit will exist and be equal to this one, should it exist:

$$\lim_{x\to\infty} \frac{1}{e^x}.$$

This limit is, of course, 0, as it is of the form $1/\infty$. It is not hard to build up from here to show that for any integer value of $n > 0$ that:

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0.$$

This is an expression of the fact that exponential functions grow faster than polynomial functions.

Examples

- What is the limit $x\log(x)$ as $x \to 0+$?

Rewriting, we see this is just:

$$\lim_{x\to 0+} \frac{\log(x)}{1/x}.$$

L'Hospital's rule clearly applies to one sided limits, as well as two (our proof sketch used one-sided limits), so this limit will equal the following, should it exist:

$$\lim_{x\to 0+} \frac{1/x}{-1/x^2} = \lim_{x\to 0+} -x = 0.$$

- What is the limit x^x as $x \to 0+$? The expression is of the form 0^0, which is indeterminate. (Even though floating point math defines the value as 1.) We can rewrite this by taking a log:

$$x^x = \exp(\log(x^x)) = \exp(x\log(x)) = \exp(\log(x)/(1/x)).$$

Be just saw that $\lim_{x\to 0+} \log(x)/(1/x) = 0$. So by the rules for limits of compositions and the fact that e^x is continuous, we see $\lim_{x\to 0+} x^x = e^0 = 1$.

- L'Hospital himself was interested in this limit for $a > 0$

$$\lim_{x\to a} \frac{\sqrt{2a^3\cdot x - x^4} - a\cdot(a^2\cdot x)^{1/3}}{a - (a\cdot x^3)^{1/4}}.$$

These derivatives can be done by hand, but to avoid any minor mistakes we utilize `SymPy` taking care to use rational numbers for the fractional powers, so as not to lose precision through floating point roundoff:

```
using CalculusWithJulia    # loads `SymPy`
using Plots
@vars a x positive=true real=true
f(x) = sqrt(2a^3*x - x^4) - a * (a^2*x)^(1//3)
g(x) = a - (a*x^3)^(1//4)
```

```
g (generic function with 1 method)
```

We can see that at $x = a$ we have the indeterminate form $0/0$:

```
f(a), g(a)
```

```
(0, 0)
```

What about the derivatives?

```
fp, gp = subs(diff(f(x),x), x=>a), subs(diff(g(x),x), x=>a)
```

```
(-4*a/3, -3/4)
```

Their ratio will not be indeterminate, so the limit in question is just the ratio:

```
fp/gp
```

$$\frac{16a}{9}$$

Of course, we could have just relied on `limit`, which knows about L'Hospital's rule:

```
limit(f(x)/g(x), x, a)
```

$$\frac{16a}{9}$$

1.3 Questions

⊛ Question
This function $f(x) = \sin(5x)/x$ is *indeterminate* at $x = 0$. What type?

1.
$$0/0$$

2.
$$\infty/\infty$$

3.
$$0^0$$

4.
$$\infty - \infty$$

5.
$$0 \cdot \infty$$

⊛ Question
This function $f(x) = \sin(x)^{\sin(x)}$ is *indeterminate* at $x = 0$. What type?

1.
$$0/0$$

2.
$$\infty/\infty$$

3.
$$0^0$$

4.
$$\infty - \infty$$

5.
$$0 \cdot \infty$$

⊛ Question
This function $f(x) = (x-2)/(x^2-4)$ is *indeterminate* at $x = 2$. What type?

1.
$$0/0$$

2.
$$\infty/\infty$$

3.
$$0^0$$

4.
$$\infty - \infty$$

5.
$$0 \cdot \infty$$

⊛ Question
This function $f(x) = (g(x+h) - g(x-h))/(2h)$ (g is continuous) is *indeterminate* at $h = 0$. What type?

1.
$$0/0$$

2.
$$\infty/\infty$$

3.
$$0^0$$

4.
$$\infty - \infty$$

5.
$$0 \cdot \infty$$

⊛ Question
This function $f(x) = x \log(x)$ is *indeterminate* at $x = 0$. What type?

1.
$$0/0$$

2.
$$\infty/\infty$$

3.
$$0^0$$

4.
$$\infty - \infty$$

5.
$$0 \cdot \infty$$

⊛ Question
Does L'Hospital's rule apply to this limit:

$$\lim_{x \to \pi} \frac{\sin(\pi x)}{\pi x}.$$

1. No. It is not indeterminate
2. Yes. It is of the form $0/0$

⊛ Question
Use L'Hospital's rule to find the limit

$$L = \lim_{x \to 0} \frac{4x - \sin(x)}{x}.$$

What is L?

⊛ Question
Use L'Hospital's rule to find the limit

$$L = \lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x}.$$

What is L?

⊛ Question
Use L'Hospital's rule *two* or more times to find the limit

$$L = \lim_{x \to 0} \frac{x - \sin(x)}{x^3}.$$

What is L?

⊛ Question
Use L'Hospital's rule *two* or more times to find the limit

$$L = \lim_{x \to 0} \frac{1 - x^2/2 - \cos(x)}{x^3}.$$

What is L?

⊛ Question
By using a common denominator to rewrite this expression, use L'Hospital's rule to find the limit

$$L = \lim_{x \to 0} \frac{1}{x} - \frac{1}{\sin(x)}.$$

What is L?

Question Use L'Hospital's rule to find the limit

$$L = \lim_{x \to \infty} \log(x)/x$$

What is L?

Question Using L'Hospital's rule, does

$$\lim_{x \to 0+} x^{\log(x)}$$

exist?

Consider $x^{\log(x)} = e^{\log(x)\log(x)}$.

1. Yes
2. No

Question Using L'Hospital's rule, find the limit of

$$\lim_{x\to 1}(2-x)^{\tan(\pi/2\cdot x)}.$$

(Hint, express as $\exp^{\tan(\pi/2\cdot x)\cdot\log(2-x)}$ and take the limit of the resulting exponent.)

1. $$e^{2/\pi}$$
2. It does not exist
3. $$2\pi$$
4. $$1$$
5. $$0$$