Solving PDEs Associated with Economic Models

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This package EconPDEs.jl introduces a fast and robust way to solve systems of PDEs + algebraic equations (i.e. DAEs) associated with economic models. This note details the underlying algorithm.

Step 1: Write Finite Difference Scheme The system of PDEs is written on a state space grid and derivatives are substituted by finite difference approximations. As in Achdou et al. (2016), first order derivatives are upwinded. This ensures that boundary counditions are satisfied, and this helps making the scheme monotonous.

Step 2: Solve Finite Difference Scheme Denote V_t the solution of the PDE. We can alwyays write the HJB is $\partial_t V = G(V_t)$. I propose to solve for V using a fully implicit Euler method. Given V_{t+1}, V_t is solved using:

$$\frac{1}{\Delta}(V_{t+1} - V_t) = G(V_t)$$

Each time step requires to solve a non-linear equation, which is solved using a Newton-Raphson method.

If the Newton-Raphson step is not successful, I decrease Δ (since this non-linear step converges if the guess is sufficiently close to the solution). If it is successful, I increase Δ, to speed up the algorithm.

This method is most similar to a method used in the fluid dynamics literature. In this context, it is called the Pseudo-Transient Continuation method, and is denoted Ψtc. Formal conditions for the convergence of the algorithm are given in Kelley and Keyes (1998).

Difference with Achdou et al. (2016) Achdou et al. (2016) focus on linear PDEs of the form

$$0 = f_1(V) + f_2(x)\partial_x V + f_3(x)\partial_{xx} V + \partial_t V$$

whereas I am interesed in semi-linear PDEs

$$0 = f_1(V) + f_2(x, \partial_x V)\partial_x V + f_3(x, \partial_x V)\partial_{xx} V + \partial_t V$$

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In this case, there is no need for non-linear step and one can solve the PDE using explicit Euler methods

References

Achdou, Yves, Jiequn Han, Jean-Michel Lasry, Pierre-Louis Lions, and Benjamin Moll, "Heterogeneous Agent Models in Continuous Time," 2016. Working Paper.

Kelley, Carl Timothy and David E Keyes, "Convergence analysis of pseudo-transient continuation," *SIAM Journal on Numerical Analysis*, 1998, *35* (2), 508–523.