Efficient Coherent States and Husimi functions in the Efficient Coherent Basis

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2020

The coefficients of a coherent state $|\boldsymbol{x}\rangle = \sum_{N=0}^{N_{\max}} \sum_{m=-j}^{j} C_{N,m}^{(\mathrm{coh})}(\boldsymbol{x}) \, |N; m, j\rangle$ in the efficient coherent basis are given by

$$C_{N,m}^{(\mathrm{coh})}(\boldsymbol{x}) = \frac{1}{\sqrt{N!}} e^{-|\alpha+Gm|^2/2} (\alpha+Gm)^N \sqrt{\binom{2j}{j+m}} \frac{w^{m+j}}{(1+|w|^2)^j}, \tag{1}$$

where $G = 2\gamma/\omega\sqrt{2j}$, $w = (1+z)/(1-z)$, $z = e^{-i\phi}\tan(\theta/2)$, $\alpha = (q+ip)\sqrt{j/2}$.

It is easy to see that

$$|C_{N,m}^{(\mathrm{coh})}(\boldsymbol{x})|^2 = \boldsymbol{P}_m(\lambda = |\alpha+Gm|^2)(N)\boldsymbol{B}(n=2j, p=\xi)(m+j)$$

where $\xi = |w|^2/(1+|w|^2)$, $\boldsymbol{P}_m(\lambda)(k) = \frac{e^{-\lambda}\lambda^k}{k!}$ is a Poisson distribution and $\boldsymbol{B}(n,p)(k) = \binom{n}{k} p^k (1-p)^{n-k}$ a binomial distribution.

Using this fact, we may know beforehand which coefficients will be small. Consider a small tolerance $0 < \varepsilon < 1$. The quantile function of a proability distribution $Q(P)$ is defined as the value of the parameter k from which the cumulative probability is greater than P, that is, if $k_i = Q(\varepsilon/2)$ and $k_f = Q(1-\varepsilon/2)$, then between k_i and k_f lies $1-\varepsilon$ of the distribution.

By letting

$$m_0 = Q_{\boldsymbol{B}}(\varepsilon/4) - j$$

$$m_f = Q_{\boldsymbol{B}}(1-\varepsilon/4) - j,$$

we will chop off the tails of the binomial distribution, throwing away $\varepsilon/2$ of the probability distribution.

Now, each Poisson distribution depends on m, so for each m between m_i and m_f, let $\varepsilon' = \varepsilon/(m_f - m_0 + 1)$[1] and

$$N_0(m) = Q_{\boldsymbol{P}_m}(\varepsilon'/4)$$

$$N_f(m) = Q_{\boldsymbol{P}_m}(1-\varepsilon'/4).$$

[1] Making $\varepsilon' = \varepsilon/(m_f - m_0 + 1)\mathbf{B}(2j,\xi)(m+j)$ works a little better, because you remove more off the Poisson distributions that attain lower values when multiplied by the Binomial distribution.

With each of these we are chopping $\varepsilon'/2$ off each Poisson distribution. This is done for each m, so in total we ignore $\varepsilon'(m_f - m_0 + 1) = \varepsilon$.

With these new bounds, we may calculate only the coefficients where m is between m_0 and m_f, and N is between $N_0(m)$ and $N_f(m)$, and he Husimi function of a state with coefficients $C_{N,M}$ is

$$\mathcal{Q}_\psi(\boldsymbol{x}) = \left| \sum_{m=m_0}^{m_f} \sum_{N=N_0(m)}^{N_f(m)} C_{N,m}^{(\mathrm{coh})}(\boldsymbol{x}) C_{N,m}^{*} \right|^2 .$$

In total, we are ignoring ε of the cumulative probability of the distributions, that is, we are taking a coherent state whose norm will be $\sim 1 - \varepsilon$. Depending on the desired numerical precision, we may change ε. Even if this number is small, doing this procedure may save a lot of time. For example, if $N_{\max}$ is big to get to high energy regions, but you take a point $\boldsymbol{x}$ at low energy, the value of $N_f(m)$ will be much smaller than $N_{\max}$, resulting in a significant reduction in computation time.