Wigner function in the efficient coherent basis (BCE)

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Consider a state in the BCE

$$|\psi\rangle = \sum_{n=0}^{Nmax} \sum_{m_x=-j}^{j} C_{n,m_x} |n; j, m_x\rangle .$$

We want the Wigner transform of the state

$$\rho = |\psi\rangle \langle\psi| = \sum_{n,n'=0}^{Nmax} \sum_{m_x,m'_x=-j}^{j} C_{n,m_x} C^*_{n',m'_x} |n; j, m_x\rangle \langle n'; j, m'_x| . \tag{1}$$

We can write it as a linear combination of the Wigner transforms of

$$\rho_{n,n',m_x,m'_x} = |n; j, m_x\rangle \langle n'; j, m'_x|.$$

1. From BCE to Fock-Dicke basis

We start by writing a state in the BCE $|n; j, m_x\rangle$ in terms of the Dicke states $|j, m_x\rangle$ and (displaced) Fock states $|n, \alpha\rangle = D(\alpha) |n\rangle$, where $D(\alpha) = \exp\left(\alpha a^\dagger - \alpha^* a\right)$. With this notation, a coherent state can be written as $|\alpha\rangle = |0, \alpha\rangle$. Using $A = a + GJ_x$ and $\alpha_{m_x} = -Gm_x$ [1], we get

$$|n; j, m_x\rangle = \frac{\left(A^\dagger\right)^n}{\sqrt{n!}} |\alpha_{m_x}\rangle |j, m_x\rangle \tag{2}$$

$$= \frac{1}{\sqrt{n!}} \sum_{k=0}^{n} \binom{n}{k} \left(a^\dagger\right)^{n-k} (GJ_x)^n D(\alpha_{m_x}) |0\rangle |j, m_x\rangle \tag{3}$$

$$= \frac{1}{\sqrt{n!}} \sum_{k=0}^{n} \binom{n}{k} \left(a^\dagger\right)^{n-k} D(\alpha_{m_x}) |0\rangle (Gm_x)^n |j, m_x\rangle . \tag{4}$$

where we used that $|j, m_x\rangle$ is an eigenstate of J_x with eigenvalue m_x. Now, we repeatedly use $\left[a^\dagger, D(\alpha_{m_x})\right] = \alpha^*_{m_x} D(\alpha_{m_x}) = \alpha_{m_x} D(\alpha_{m_x})$ along with the commutation rules. We get

$$\left(a^\dagger\right)^{n-k} D(\alpha_{m_x}) = \sum_{l=0}^{n-k} \alpha_{m_x}^{n-k-l} \binom{n-k}{l} D(\alpha_{m_x}) \left(a^\dagger\right)^l . \tag{5}$$

Introducing this result above, and remembering that $\left(a^\dagger\right)^l |0\rangle = \sqrt{l!}\, |l\rangle$,

$$|n; j, m_x\rangle = \frac{1}{\sqrt{n!}} \sum_{k=0}^{n} \binom{n}{k} \sum_{l=0}^{n-k} \alpha_{m_x}^{n-k-l} \frac{(n-k)!}{l!(n-k-l)!} D(\alpha_{m_x}) \sqrt{l!}\, |l\rangle (Gm_x)^k |j, m_x\rangle \tag{6}$$

$$= \sqrt{n!} \sum_{k=0}^{n} \sum_{l=0}^{n-k} \frac{\alpha_{m_x}^{n-l} (-1)^k}{\sqrt{l!}(n-k-l)!k!} |l, \alpha_{m_x}\rangle |j, m_x\rangle \tag{7}$$

$$= \sqrt{n!} \sum_{l=0}^{n} \sum_{k=0}^{n-l} \frac{\alpha_{m_x}^{n-l} (-1)^k}{\sqrt{l!}(n-k-l)!k!} |l, \alpha_{m_x}\rangle |j, m_x\rangle \tag{8}$$

$$= \sqrt{n!} \sum_{l=0}^{n} \frac{\alpha_{m_x}^{n-l}}{\sqrt{l!}} \left(\sum_{k=0}^{n-l} \frac{(-1)^k}{k!(n-k-l)!} \right) |l, \alpha_{m_x}\rangle |j, m_x\rangle \tag{9}$$

It turns out that $\sum_{k=0}^{n-l} \frac{(-1)^k}{k!(n-k-l)!} = \delta_{n,l}$, so

$$|n; j, m_x\rangle = |n, \alpha_{m_x}\rangle\, |j, m_x\rangle\,. \tag{10}$$

2. Wigner function in the displaced Fock basis

The Wigner function for the product of Fock states $|n\rangle \langle n'|$ is [2] (Eq. 2.5)

$$\begin{aligned} W_{|n\rangle\langle n'|}(q,p) =& \sqrt{\frac{n'!}{n!}} e^{i(n'-n)\arctan(p/q)} \frac{(-1)^{n'}}{\pi} \left(2\left(q^2+p^2\right)\right)^{(n'-n)/2} \times \\ & \times L_{n'}^{n-n'}\left(2\left(q^2+p^2\right)\right) e^{-\left(q^2+p^2\right)}, \end{aligned} \tag{11}$$

where L_a^b is the associated Laguerre polynomial. We have displaced Fock states, but note that in general, for a displaced operator $\rho_{\beta,\beta'} = D(\beta)\rho D(\beta')$, where β is real (as is the case for $\beta = \alpha_{m_x} = -Gm_x$), the Wigner function is

$$\begin{aligned} W_{\rho_{\beta,\beta'}}(q,p) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \langle q+r|\, D(\beta)\rho D(\beta')\, |q-r\rangle\, e^{-2ipr}\, dr && (12) \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \langle q+r-\beta|\, \rho\, \left|q-r+\beta'\right\rangle e^{-2ipr}\, dr && (13) \end{aligned}$$

with $\hbar = 1$. Performing the change of variable $r' = r - \frac{\beta'+\beta}{2}$ y $q' = q + \frac{\beta'-\beta}{2}$, we obtain

$$\begin{aligned} W_{\rho_{\beta,\beta'}}(q,p) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \left\langle q'+r'\right|\, \rho\, \left|q'-r'\right\rangle e^{-2ipr'} e^{-ip(\beta+\beta')}\, dr' && (14) \\ &= e^{-ip(\beta+\beta')} \frac{1}{\pi} \int_{-\infty}^{\infty} \left\langle q'+r'\right|\, \rho\, \left|q'-r'\right\rangle e^{-2ipr'}\, dr' && (15) \\ &= e^{-ip(\beta+\beta')} W_\rho(q',p). && (16) \end{aligned}$$

In the particular case of the states $|n,\alpha_{m_x}\rangle \left\langle n', \alpha_{m'_x}\right| = D(\alpha_{m_x})\,|n\rangle \left\langle n'\right| D(-\alpha_{m'_x})$, we have $\beta = \alpha_{m_x}$ and $\beta' = -\alpha_{m'_x}$, so the change of variables is $q' = q + G(m_x + m'_x)/2$, and $\beta + \beta' = G(m'_x - m_x)$.

3. Wigner function in the Dicke Basis

For the product of the Dicke states $|j,m\rangle \left\langle j,m'\right|$, the Wigner function is [3]

$$W_{|j,m\rangle\langle j,m'|}(\theta,\phi) = \sum_{k=0}^{2j} \sum_{s=-k}^{k} (-1)^{j-m} \sqrt{2k+1} \begin{pmatrix} j & k & j \\ -m & s & m' \end{pmatrix} Y_{k\,s}(\theta,\phi), \tag{17}$$

where $Y_{k\,s}$ are the sherical harmonics and $\begin{pmatrix} j & k & j \\ -m & s & m' \end{pmatrix}$ the 3-j Wigner symbols.

Because we work with m_x and not m, we have to rotate $R_{x\to z} : J_x \mapsto J_z$ in the variables θ, ϕ before evaluating. That is, we have to replace $\theta, \phi \to R_{x\to z}(\theta,\phi)$.

4. Total Wigner function

With all the above, we can write the Wigner of Eq. (1) as

$$\begin{aligned} W_\rho(q,p,\theta,\phi) =& \sum_{n,n'=0}^{Nmax} \sum_{m_x,m'_x=-j}^{j} C_{n,m_x} C^*_{n',m'_x} \times \\ & \times e^{-ipG(m'_x-m_x)} W_{|n\rangle\langle n'|}(q + G(m_x+m'_x)/2, p) W_{|j,m\rangle\langle j,m'|}(R_{x\to z}(\theta,\phi)) \end{aligned} \tag{18}$$

by substituting Eqs. (11) and (17).

5. Projection in q, p

Let us compute $W_\rho(\theta,\phi) = \int\int W_\rho(q,p,\theta,\phi)\mathrm{d}q\mathrm{d}p$. For that, we must first compute the integrals of the Wigner function of the displaced Fock states. This is straightforward using the trace property of the Wigner functions, allong with the expression for the overlap of the displaced Fock states [4]

$$
\begin{aligned}
&\int\int W_{|n,\alpha_{m_x}\rangle\langle n',\alpha_{m'_x}|}(q,p)\mathrm{d}q\mathrm{d}p \\
&= \mathrm{tr}\left(|n,\alpha_{m_x}\rangle\langle n',\alpha_{m'_x}|\right) \\
&= \langle n',\alpha_{m'_x}|n,\alpha_{m_x}\rangle \\
&= \sqrt{\frac{n'!}{n!}}(\alpha_m-\alpha'_m)^{n-n'}e^{-\frac{1}{2}|\alpha_m-\alpha'_m|^2}L_{n'}^{n-n'}\left(|\alpha_m-\alpha'_m|^2\right)
\end{aligned}
$$

Then,

$$
\begin{aligned}
\int\int W_\rho(q,p,\theta,\phi)\mathrm{d}q\mathrm{d}p = \sum_{n,n'=0}^{Nmax}\sum_{m_x,m'_x=-j}^{j} C_{n,m_x}C^*_{n',m'_x}\times \\
\times\sqrt{\frac{n'!}{n!}}(\alpha_m-\alpha'_m)^{n-n'}e^{-\frac{1}{2}|\alpha_m-\alpha'_m|^2}L_{n'}^{n-n'}\left(|\alpha_m-\alpha'_m|^2\right)W_{|j,m\rangle\langle j,m'|}(R_{x\to z}(\theta,\phi)).
\end{aligned}
\tag{19}
$$

6. Projection in θ, ϕ

Let's compute $W_\rho(p,q) = \int\int W_\rho(q,p,\theta,\phi)\mathrm{d}\theta\mathrm{d}\phi$. As before,

$$
\begin{aligned}
&\int\int W_{|j,m_x\rangle\langle j,m'_x|}(\theta,\phi)\mathrm{d}\theta\mathrm{d}\phi \\
&= \mathrm{tr}\left(|j,m_x\rangle\langle j,m'_x|\right) \\
&= \langle j,m'_x|j,m_x\rangle = \delta_{m_x,m'_x}
\end{aligned}
$$

Then,

$$
\int\int W_\rho(q,p,\theta,\phi)\mathrm{d}\theta\mathrm{d}\phi = \sum_{n,n'=0}^{Nmax}\sum_{m_x=-j}^{j} C_{n,m_x}C^*_{n',m_x}W_{|n\rangle\langle n'|}(q+Gm_x,p). \tag{20}
$$

References

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